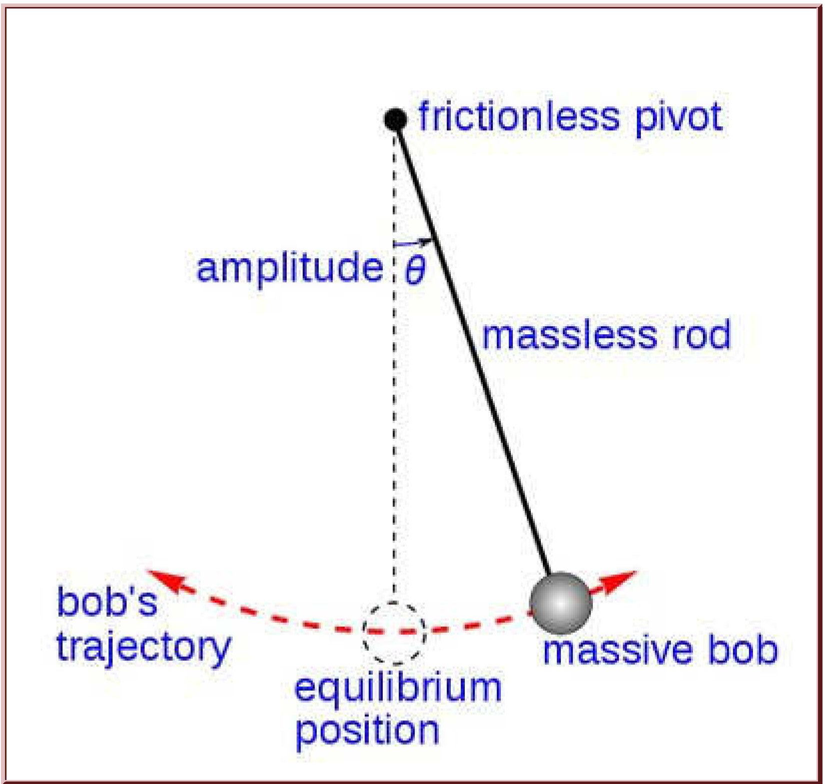
Circular Error in a Pendulum

What *causes* circular error? A search on Google provided many links to articles on the subject, but I could not find any that said "why." This page is designed to answer that question in a reader-friendly manner: if you want to, you can scroll down when you get to the maths, skip the maths, go to the graphs and continue reading. This page is also designed for readers with smartphones, which is why some of the images are presented smaller than original.

There are numerous forces acting upon a pendulum as it swings back and forth in a mechanical clock, such as gravity, elastic energy from the suspension spring, and air resistance. On this page, the subject is gravity. It is well known among clockmakers that when the angle of swing changes, the accuracy of the timekeeping changes in clocks. The effect of gravity upon timekeeping accuracy can be measured by finding how long it takes for the pendulum to go over and back. The time it takes to go over and back is known as the period.



The moment a pendulum is released, gravity pulls it straight down, and a portion of that pull acts to pull the pendulum towards its vertical position, depending on the angle, the length of the pendulum, and the force of gravity, given by the following equation, in which acceleration is theta with an umlaut (two dots) over it:

$$\ddot{\theta} + \frac{g}{l} \sin\theta = 0$$

In a computer spreadsheet, the acceleration acting upon the pendulum can therefore be calculated for each moment in time, a small fraction of a second. Then the speed (angular velocity) of the pendulum is found by multiplying the acceleration by the amount of time (a small fraction of a second). The distance traveled (arc) in that amount of time is found by multiplying the angular velocity by the amount of time. The angle in radians is found by dividing the distance by the length of the pendulum. Then the angle in degrees is found by multiplying the angle in radians by 180 and dividing by pi (3.14). The results come together in a chart like this one.

6	time	2	period in seconds				
7	0.0001	0.993621	metres length		9.80665	gravity	
8	time	acc	vel	angle	radians	d	sinΘ
9							
10	0	2.220178	0	13	0.226893	0.225446	0.224951
11	0.0001	2.220178	0.000222	13	0.226893	0.225446	0.224951
12	0.0002	2.220177	0.000444	13	0.226893	0.225445	0.224951
13	0.0003	2.220177	0.000666	12.99999	0.226893	0.225445	0.224951
14	0.0004	2.220176	0.000888	12.99999	0.226893	0.225445	0.224951
15	0.0005	2.220175	0.00111	12.99998	0.226892	0.225445	0.224951
16	0.0006	2.220173	0.001332	12.99997	0.226892	0.225445	0.224951
17	0.0007	2.220172	0.001554	12.99996	0.226892	0.225445	0.22495
18	0.0008	2.22017	0.001776	12.99995	0.226892	0.225445	0.22495
19	0.0009	2.220168	0.001998	12.99994	0.226892	0.225445	0.22495
20	0.001	2.220166	0.00222	12.99993	0.226892	0.225444	0.22495
9998	0.9988	-2.22016	0.002714	-12.9999	-0.22689	-0.22544	-0.22495
9999	0.9989	-2.22017	0.002492	-12.9999	-0.22689	-0.22544	-0.22495
10000	0.999	-2.22017	0.00227	-12.9999	-0.22689	-0.22544	-0.22495
10001	0.9991	-2.22017	0.002048	-13	-0.22689	-0.22544	-0.22495
10002	0.9992	-2.22017	0.001826	-13	-0.22689	-0.22544	-0.22495
10003	0.9993	-2.22017	0.001604	-13	-0.22689	-0.22545	-0.22495
10004	0.9994	-2.22017	0.001382	-13	-0.22689	-0.22545	-0.22495
10005	0.9995	-2.22018	0.00116	-13	-0.22689	-0.22545	-0.22495
10006	0.9996	-2.22018	0.000938	-13	-0.22689	-0.22545	-0.22495
10007	0.9997	-2.22018	0.000716	-13	-0.22689	-0.22545	-0.22495
10008	0.9998	-2.22018	0.000494	-13	-0.22689	-0.22545	-0.22495
10009	0.9999	-2.22018	0.000272	-13	-0.22689	-0.22545	-0.22495
10010	1	-2.22018	4.96E-05	-13	-0.22689	-0.22545	-0.22495
10011	1.0001	-2.22018	-0.00017	-13	-0.22689	-0.22545	-0.22495
10012	1.0002	-2.22018	-0.00039	-13	-0.22689	-0.22545	-0.22495
10013	1.0003	-2.22018	-0.00062	-13	-0.22689	-0.22545	-0.22495
10014	1.0004	-2.22018	-0.00084	-13	-0.22689	-0.22545	-0.22495
10015	1.0005	-2.22017	-0.00106	-13	-0.22689	-0.22545	-0.22495
10016	1.0006	-2.22017	-0.00128	-13	-0.22689	-0.22545	-0.22495
10017	1.0007	-2.22017	-0.0015	-13	-0.22689	-0.22544	-0.22495
10018	1.0008	-2.22017	-0.00173	-13	-0.22689	-0.22544	-0.22495
10019	1.0009	-2.22017	-0.00195	-12.9999	-0.22689	-0.22544	-0.22495
10020	1.001	-2.22017	-0.00217	-12.9999	-0.22689	-0.22544	-0.22495

The formula in each cell of the spreadsheet is written so that the angle can be changed and the rest of the chart will be re-calculated to find how long it takes for the pendulum to reach the other side. In this chart, the angle shown is 13° and it takes one second for the pendulum to reach the other side. That is expected because a one-second pendulum is 39.1 inches (0.994 metres) long. When the pendulum reaches the other side, it changes direction to swing back, and this is seen in the chart when the velocity becomes negative. You can click on the chart below to see it full size: make your own chart and experiment with it.

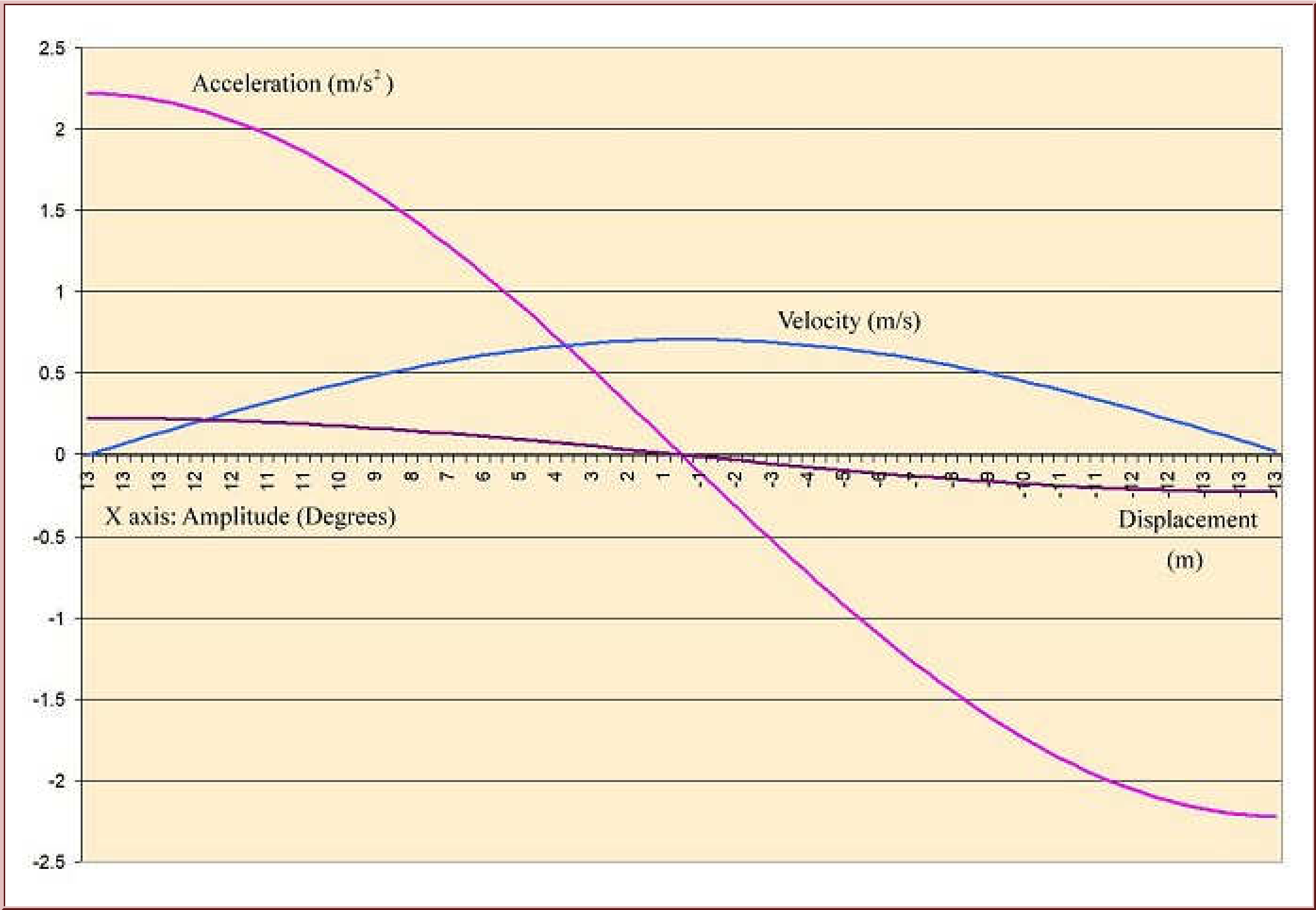
	A	B	C	D	E	F	G
1							
2							
3							
4							
5							
6	time	2	period in seconds				
7	0.0001	=E7*(((B6/(2*PI()))^2))	metres length		9.80665	gravity	
8	time	acceleration	velocity	angle	radians	distance	sinΘ
9							
10	0	=($E$7/$B$7)*G10	0	13	=(D10*PI())/180	=B$7*E10	=SIN(E10)
11	=A10+A$7	=($E$7/$B$7)*G11	=C10+(B10*A$7)	=(E11*180)/PI()	=F11/B$7	=F10-(C11*A$7)	=SIN(E11)
12	=A11+A$7	=($E$7/$B$7)*G12	=C11+(B11*A$7)	=(E12*180)/PI()	=F12/B$7	=F11-(C12*A$7)	=SIN(E12)
13	=A12+A$7	=($E$7/$B$7)*G13	=C12+(B12*A$7)	=(E13*180)/PI()	=F13/B$7	=F12-(C13*A$7)	=SIN(E13)
14	=A13+A$7	=($E$7/$B$7)*G14	=C13+(B13*A$7)	=(E14*180)/PI()	=F14/B$7	=F13-(C14*A$7)	=SIN(E14)
15	=A14+A$7	=($E$7/$B$7)*G15	=C14+(B14*A$7)	=(E15*180)/PI()	=F15/B$7	=F14-(C15*A$7)	=SIN(E15)
16	=A15+A$7	=($E$7/$B$7)*G16	=C15+(B15*A$7)	=(E16*180)/PI()	=F16/B$7	=F15-(C16*A$7)	=SIN(E16)
17	=A16+A$7	=($E$7/$B$7)*G17	=C16+(B16*A$7)	=(E17*180)/PI()	=F17/B$7	=F16-(C17*A$7)	=SIN(E17)
18	=A17+A$7	=($E$7/$B$7)*G18	=C17+(B17*A$7)	=(E18*180)/PI()	=F18/B$7	=F17-(C18*A$7)	=SIN(E18)
19	=A18+A$7	=($E$7/$B$7)*G19	=C18+(B18*A$7)	=(E19*180)/PI()	=F19/B$7	=F18-(C19*A$7)	=SIN(E19)

When the angle is changed to 45°, it takes 1.0367 seconds for the pendulum to swing from one side to the other, almost 4% longer. Since time taken becomes longer, a clock would be seen to be losing time.

6	time	2	period in seconds				
7	0.0001	0.993621	metres length		9.80665	gravity	
8	time	acc	vel	angle	radians	d	sinΘ
9							
10	0	6.978864	0	45	0.785398	0.780388	0.707107
11	0.0001	6.978864	0.000698	45	0.785398	0.780388	0.707107
12	0.0002	6.978863	0.001396	44.99999	0.785398	0.780388	0.707107
13	0.0003	6.978861	0.002094	44.99998	0.785398	0.780388	0.707106
14	0.0004	6.978859	0.002792	44.99996	0.785397	0.780388	0.707106
15	0.0005	6.978857	0.003489	44.99994	0.785397	0.780387	0.707106
16	0.0006	6.978854	0.004187	44.99992	0.785397	0.780387	0.707106
17	0.0007	6.97885	0.004885	44.99989	0.785396	0.780386	0.707105
18	0.0008	6.978847	0.005583	44.99986	0.785396	0.780386	0.707105
19	0.0009	6.978842	0.006281	44.99982	0.785395	0.780385	0.707105
20	0.001	6.978837	0.006979	44.99978	0.785394	0.780385	0.707104
10364	1.0354	-6.97883	0.008732	-44.9997	-0.78539	-0.78038	-0.7071
10365	1.0355	-6.97883	0.008034	-44.9998	-0.78539	-0.78038	-0.7071
10366	1.0356	-6.97884	0.007336	-44.9998	-0.78539	-0.78038	-0.7071
10367	1.0357	-6.97884	0.006639	-44.9998	-0.7854	-0.78039	-0.7071
10368	1.0358	-6.97885	0.005941	-44.9999	-0.7854	-0.78039	-0.70711
10369	1.0359	-6.97885	0.005243	-44.9999	-0.7854	-0.78039	-0.70711
10370	1.036	-6.97886	0.004545	-44.9999	-0.7854	-0.78039	-0.70711
10371	1.0361	-6.97886	0.003847	-44.9999	-0.7854	-0.78039	-0.70711
10372	1.0362	-6.97886	0.003149	-45	-0.7854	-0.78039	-0.70711
10373	1.0363	-6.97886	0.002451	-45	-0.7854	-0.78039	-0.70711
10374	1.0364	-6.97886	0.001753	-45	-0.7854	-0.78039	-0.70711
10375	1.0365	-6.97886	0.001055	-45	-0.7854	-0.78039	-0.70711
10376	1.0366	-6.97886	0.000358	-45	-0.7854	-0.78039	-0.70711
10377	1.0367	-6.97886	-0.00034	-45	-0.7854	-0.78039	-0.70711
10378	1.0368	-6.97886	-0.00104	-45	-0.7854	-0.78039	-0.70711
10379	1.0369	-6.97886	-0.00174	-45	-0.7854	-0.78039	-0.70711
10380	1.037	-6.97886	-0.00243	-45	-0.7854	-0.78039	-0.70711
10381	1.0371	-6.97886	-0.00313	-45	-0.7854	-0.78039	-0.70711
10382	1.0372	-6.97886	-0.00383	-44.9999	-0.7854	-0.78039	-0.70711
10383	1.0373	-6.97885	-0.00453	-44.9999	-0.7854	-0.78039	-0.70711

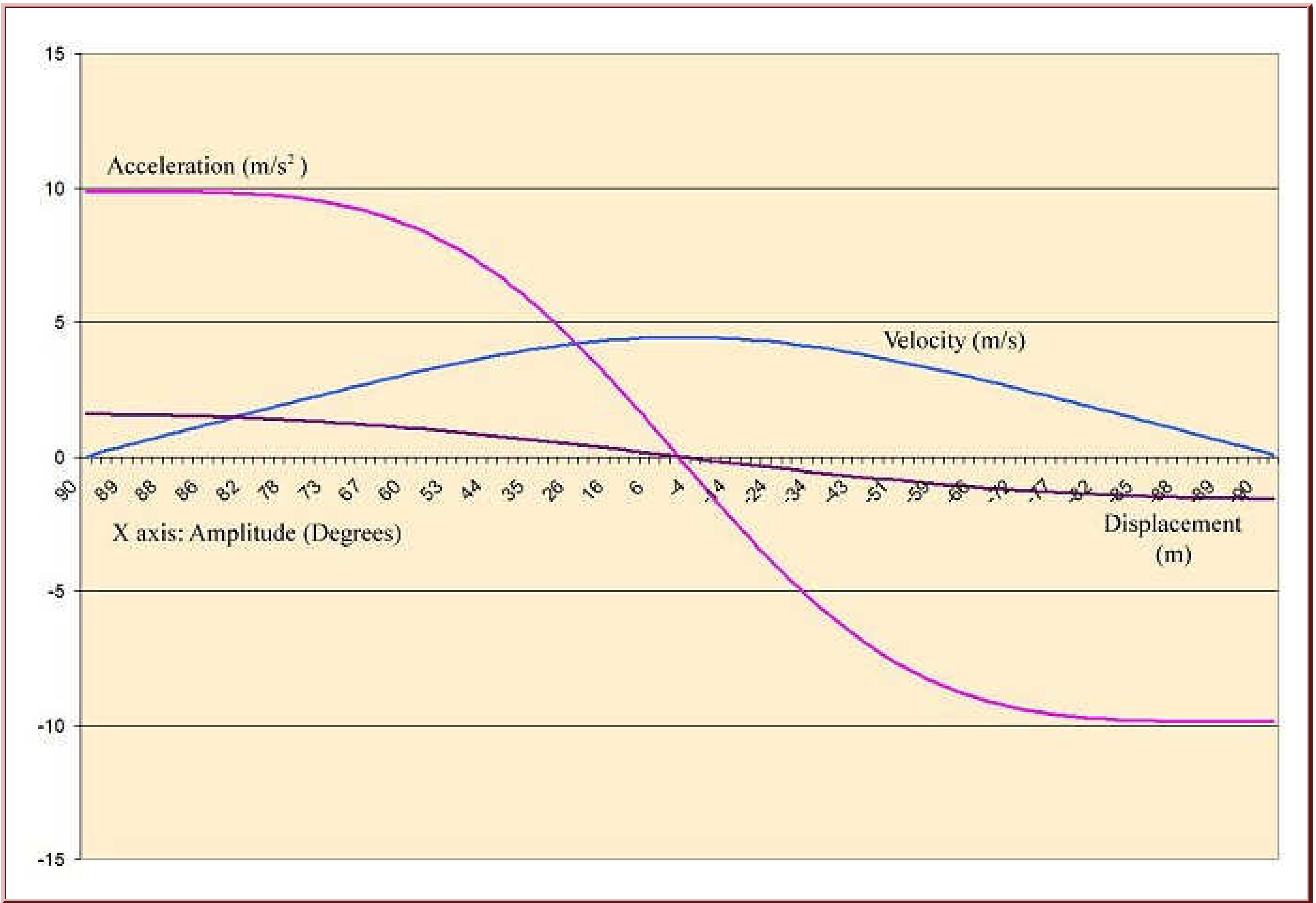
The position of the pendulum (its angle) affects how gravity acts upon it, and the result is that the acceleration changes continuously. You can click on each one of all the graphs below to see it full size.

For a 13° arc from vertical:

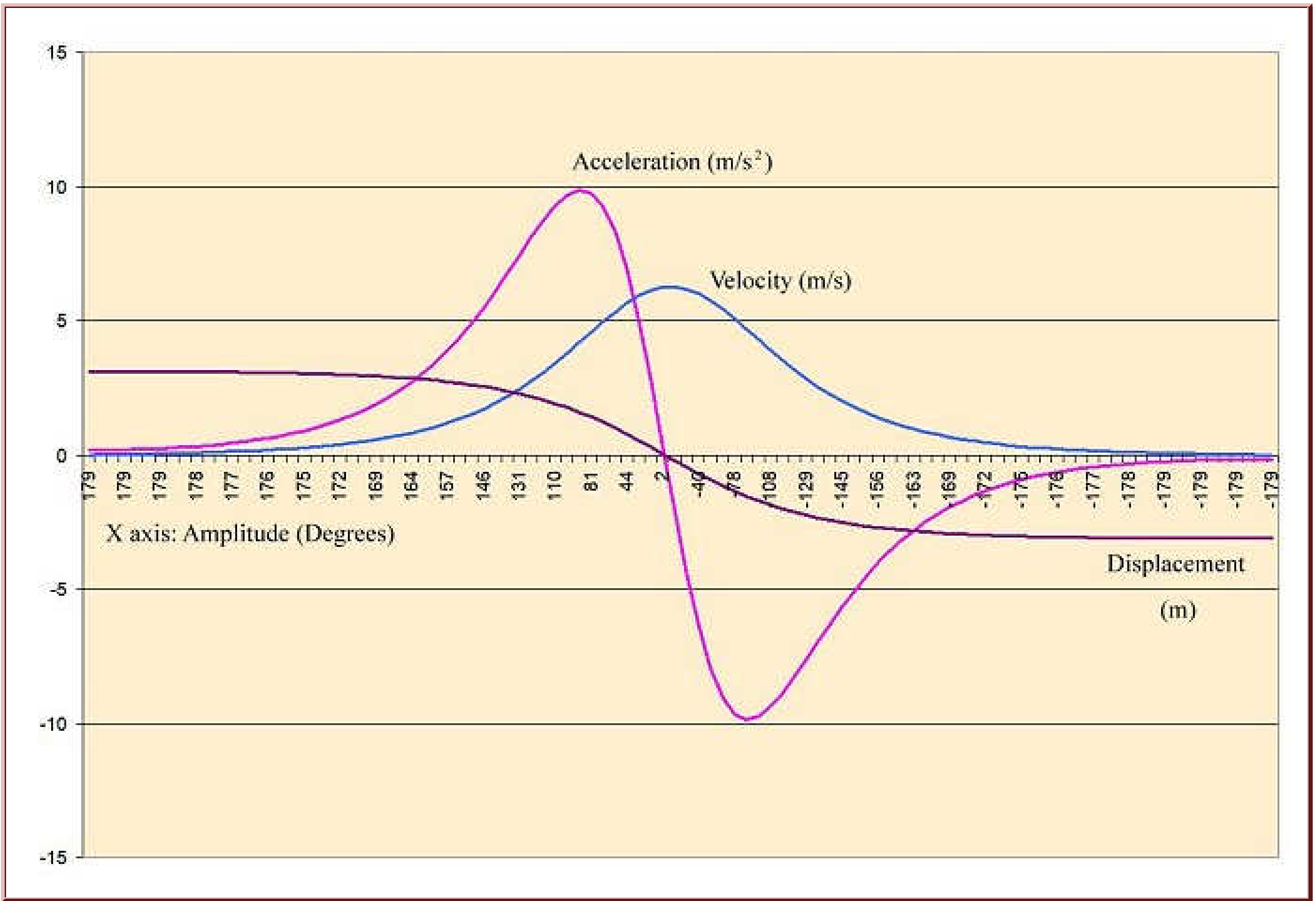


The effect becomes more obvious for large angles.

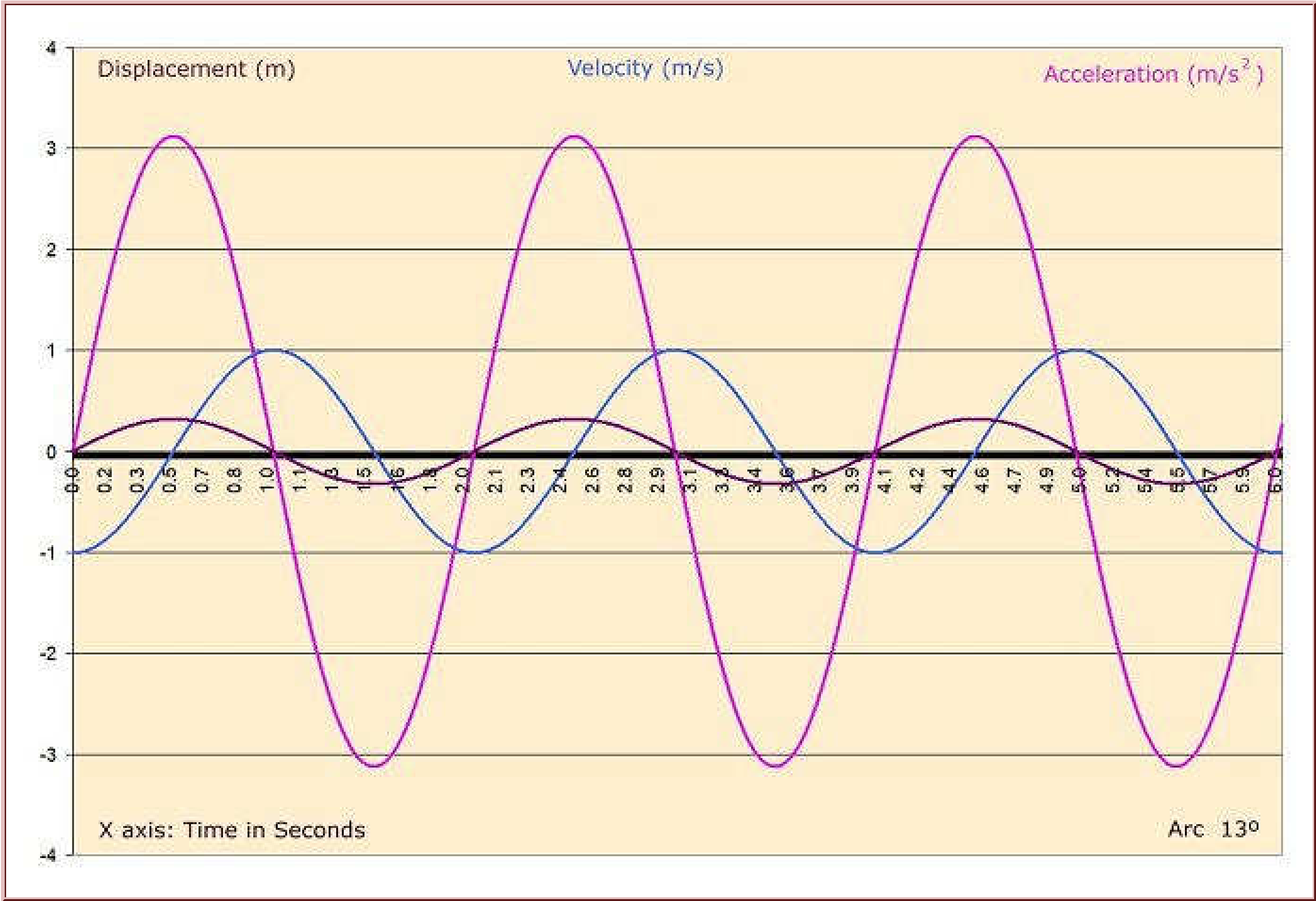
For a 90° arc from vertical:

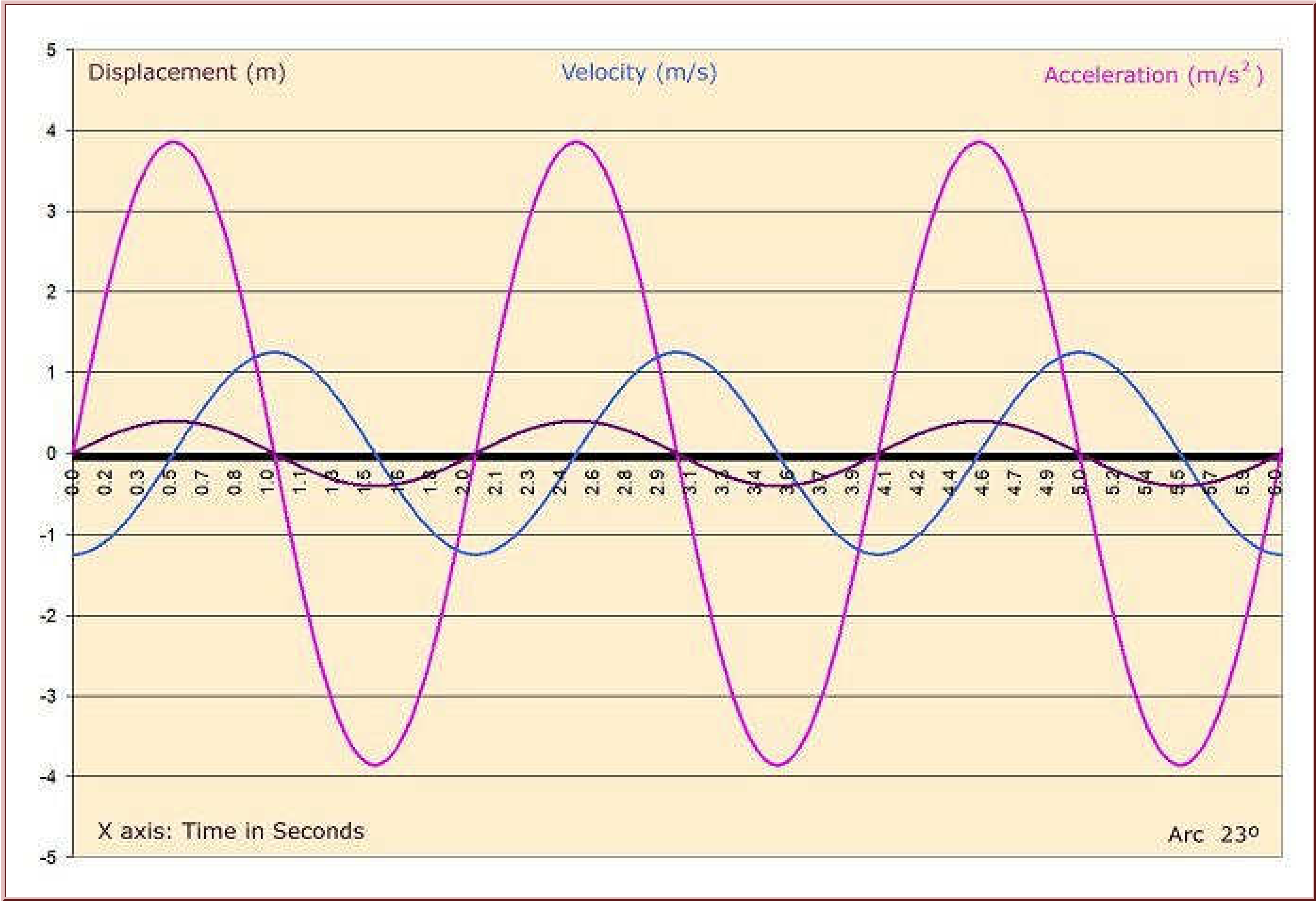


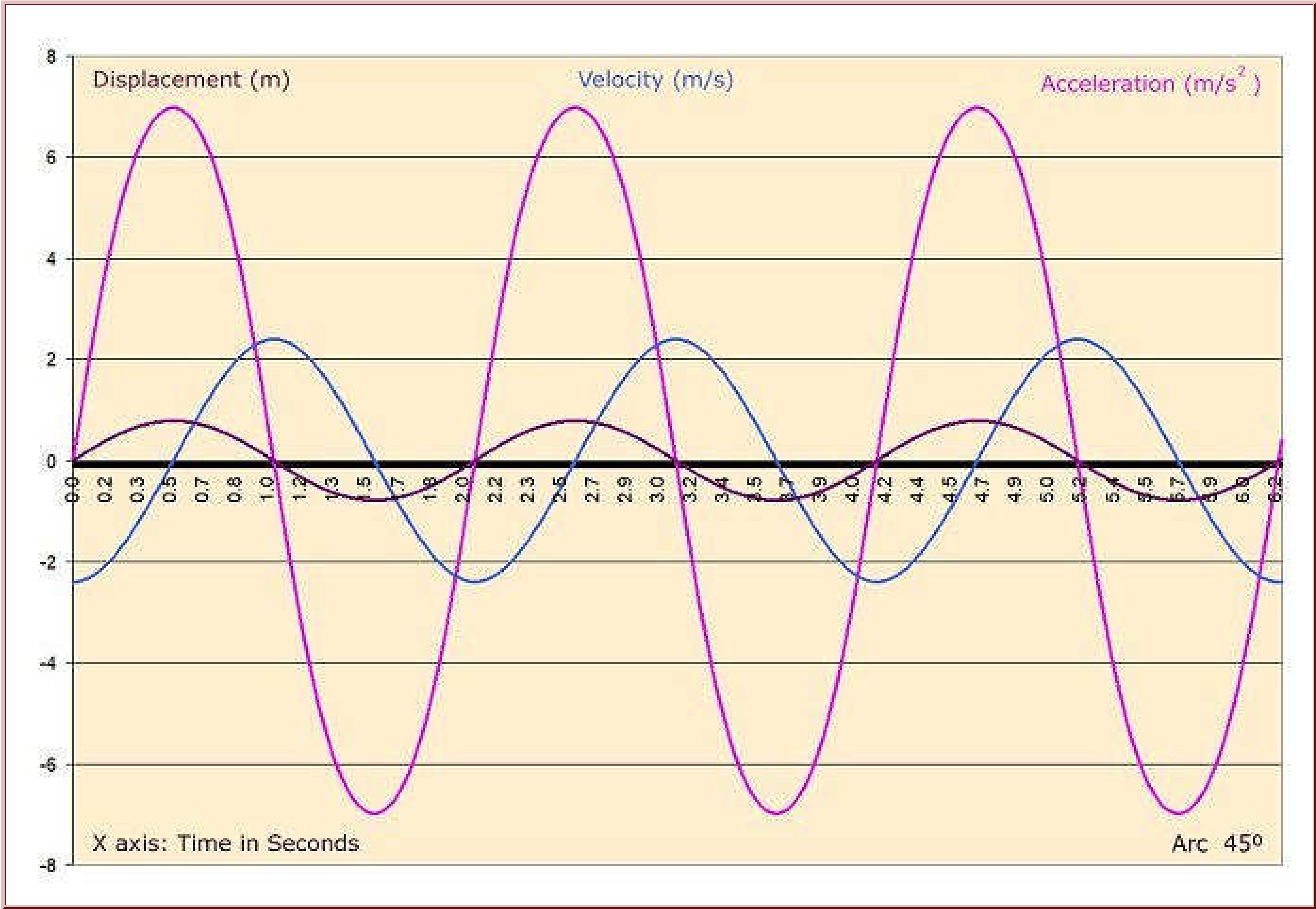
At 180°, the period goes to infinity.
For a 179° arc from vertical:

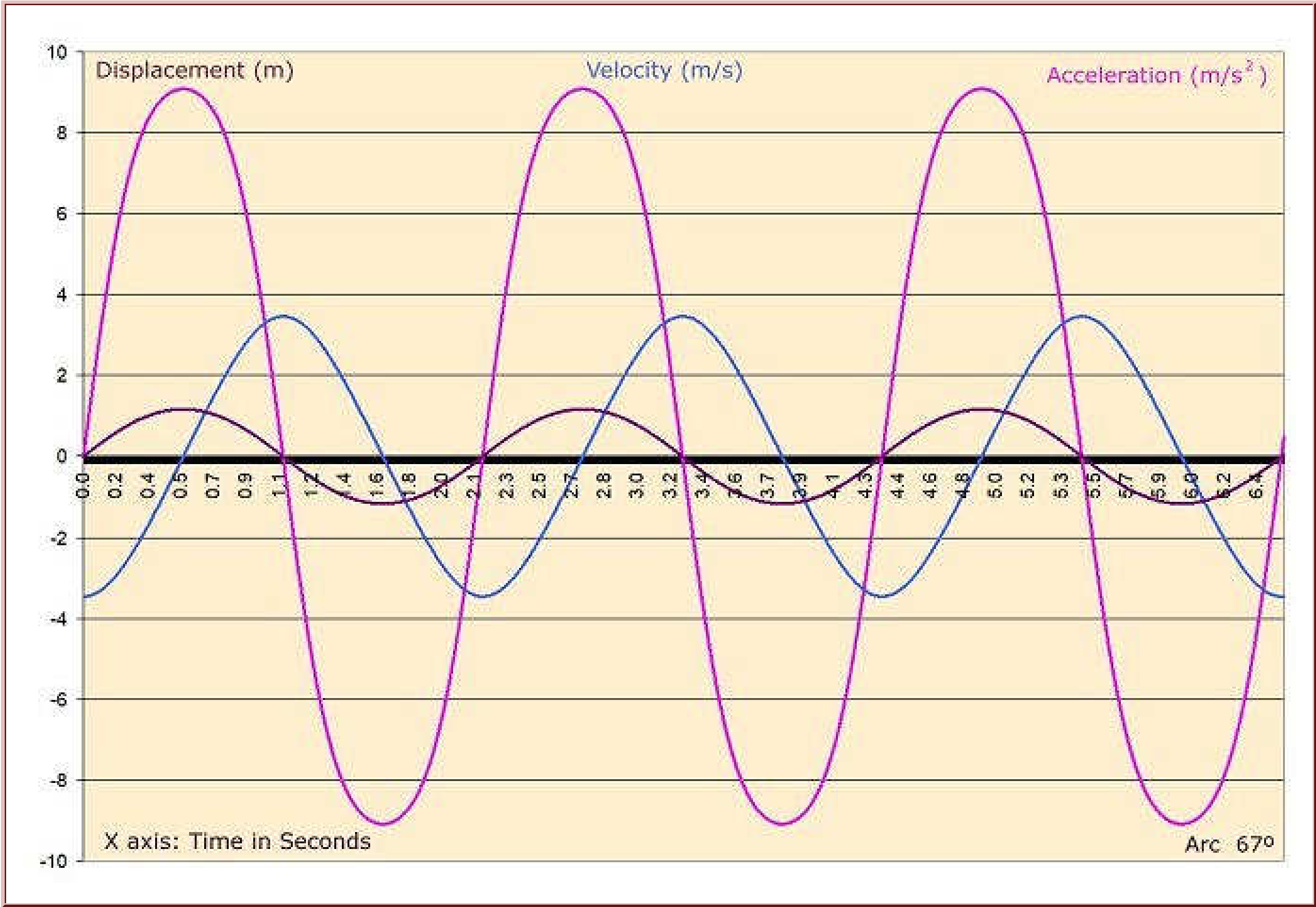


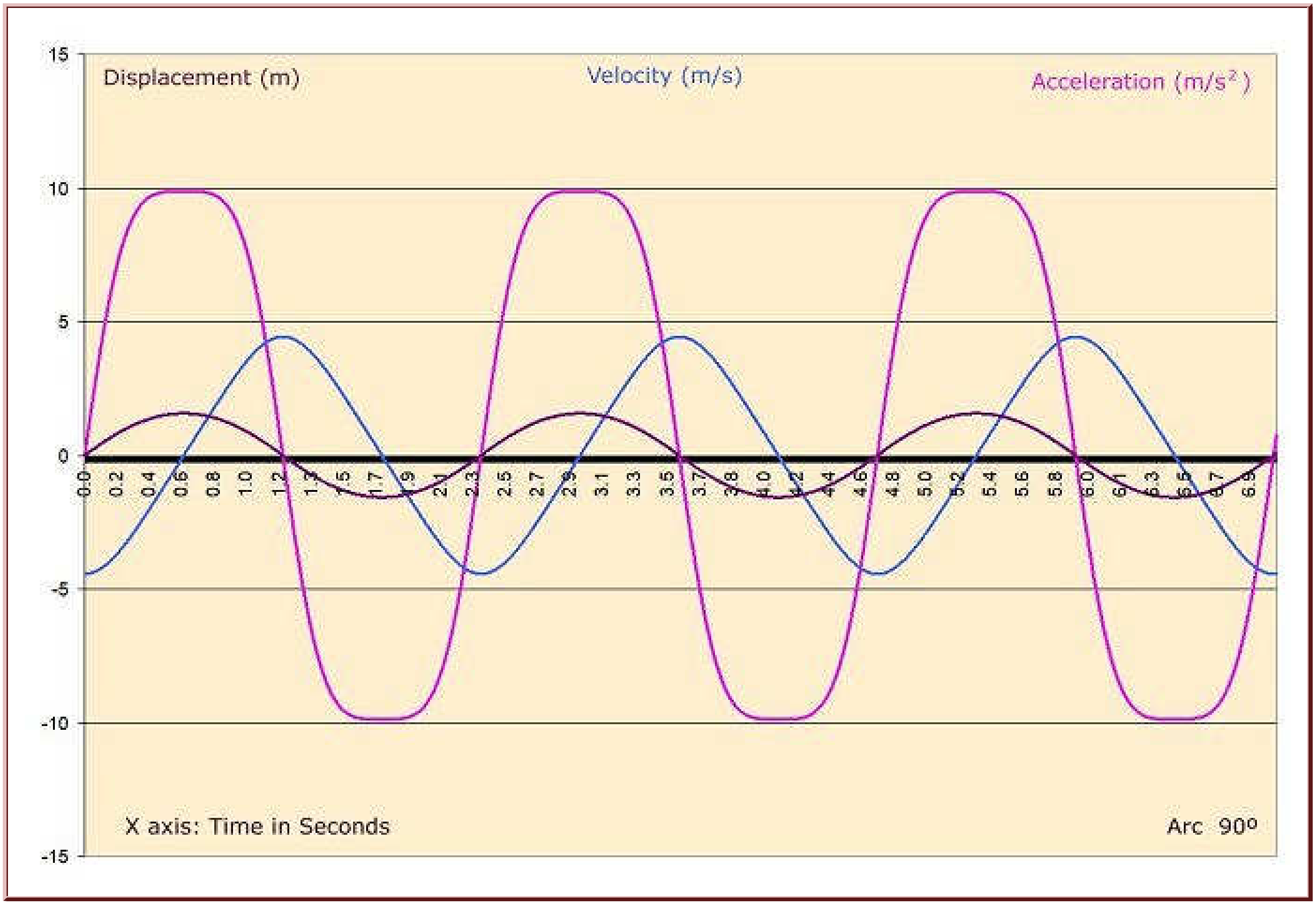
Looking at several graphs below, we can see that, as the angle increases, the height (amplitude) of the acceleration curve increases, but at a declining rate: when the angle doubles, the acceleration increases by less than double.


Displacement (m)
Velocity (m/s)
Acceleration (m/s²)
X axis: Time in Seconds
Arc 13º

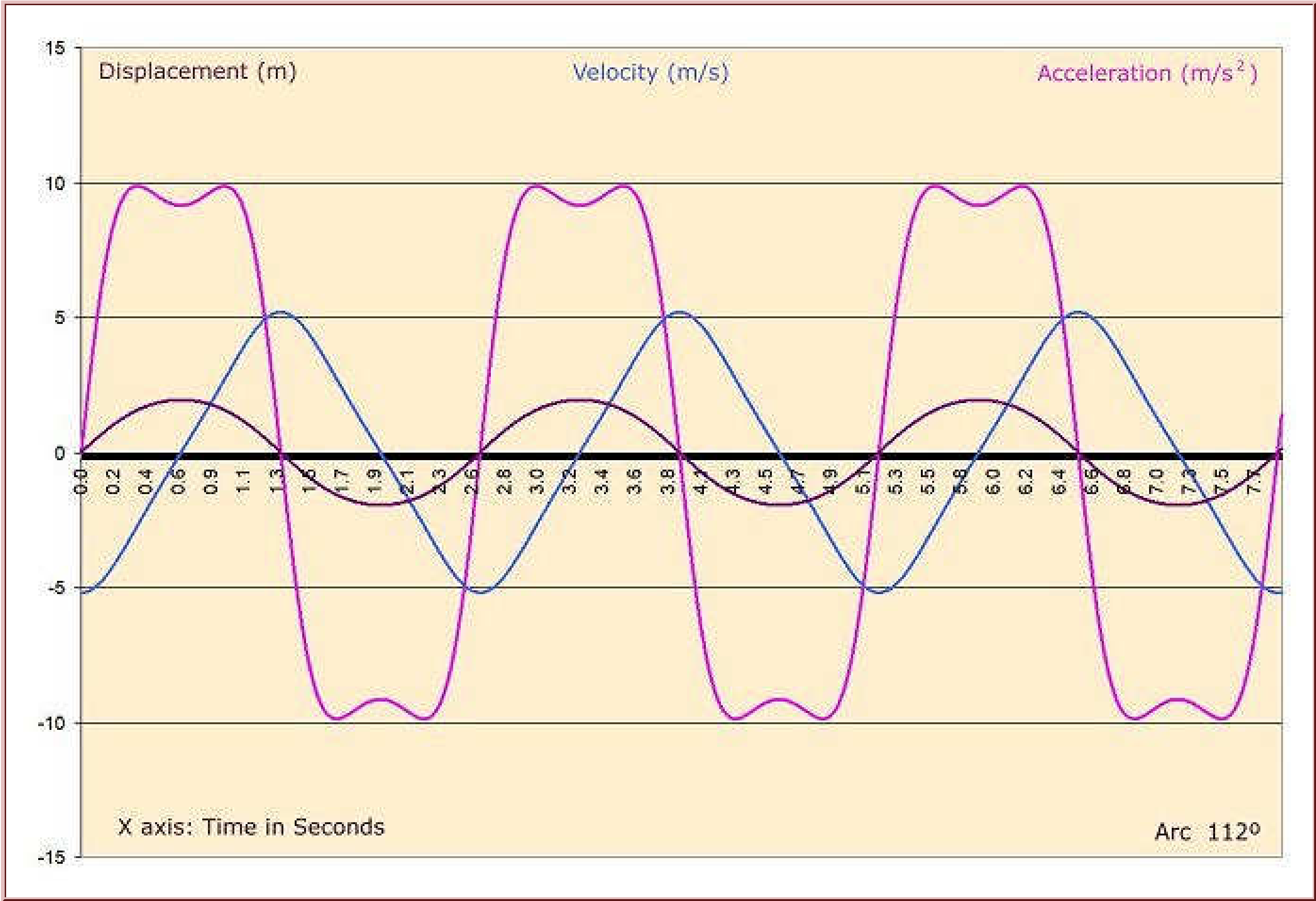

Displacement (m)
Velocity (m/s)
Acceleration (m/s²)
X axis: Time in Seconds
Arc 23º


Displacement (m)
Velocity (m/s)
Acceleration (m/s²)
X axis: Time in Seconds
Arc 45º
8
6
4
2
0
-2
-4
-6
-8

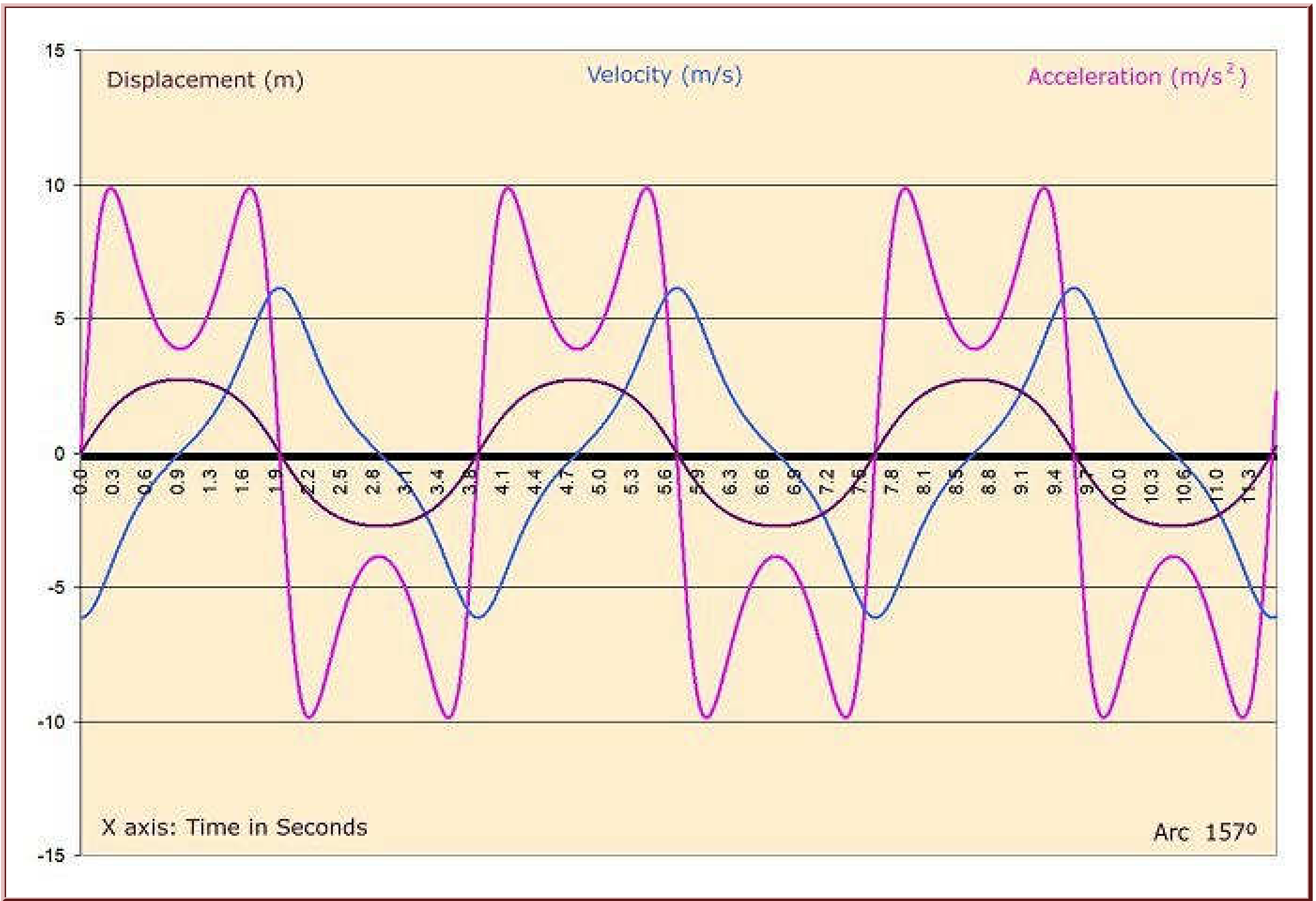

Displacement (m)
Velocity (m/s)
Acceleration (m/s^2)
X axis: Time in Seconds
Arc 67º



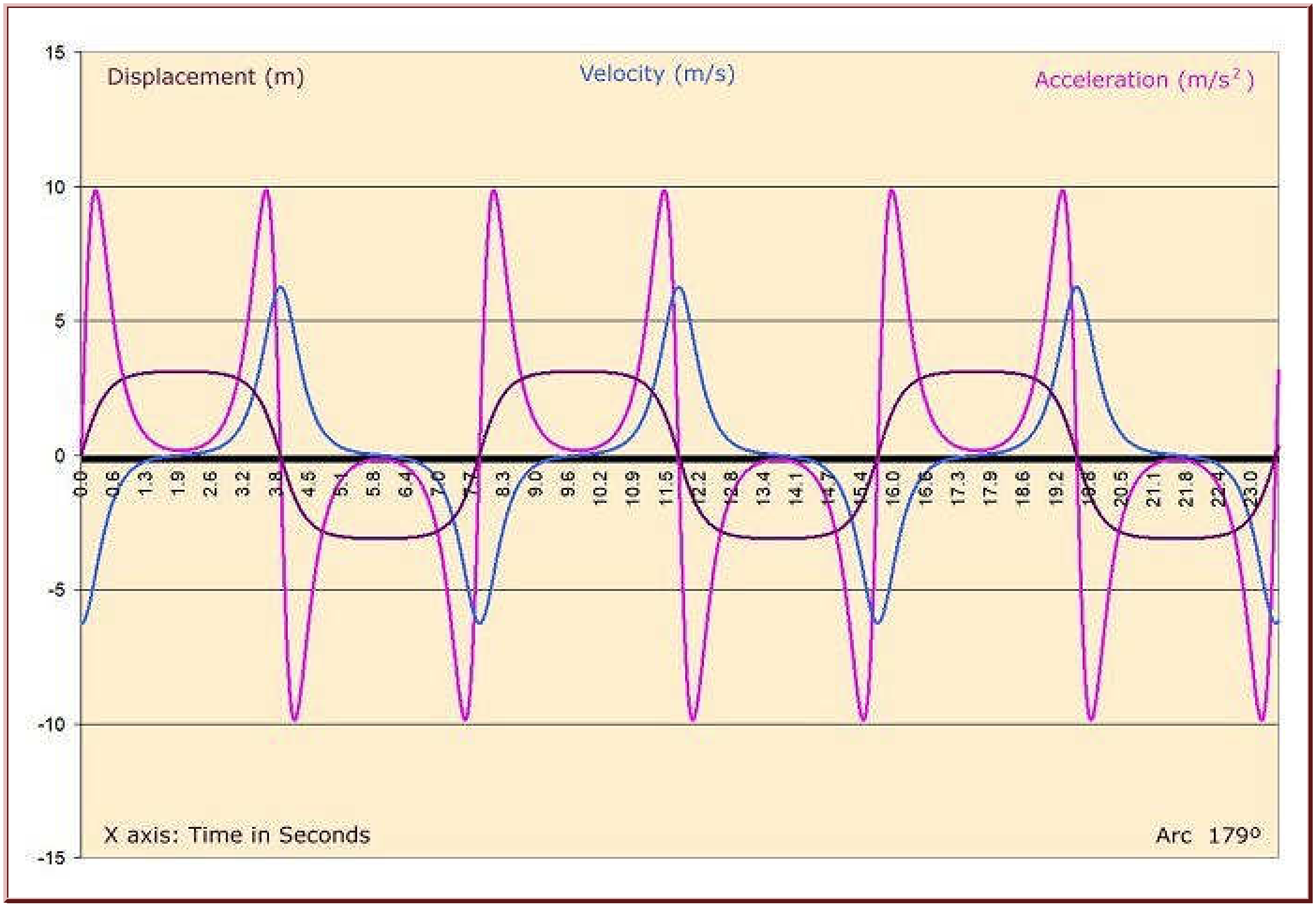
Beyond 90°, the acceleration curve no longer reaches a higher high, and forms what looks like a wave within a wave.


Displacement (m)
Velocity (m/s)
Acceleration (m/s²)
X axis: Time in Seconds
Arc 112º


Displacement (m)
Velocity (m/s)
Acceleration (m/s²)
15
10
5
0
-5
-10
-15
0.0 0.2 0.5 0.7 1.0 1.2 1.5 1.7 2.0 2.2 2.5 2.7 3.0 3.2 3.5 3.7 4.0 4.2 4.5 4.7 5.0 5.2 5.5 5.7 6.0 6.2 6.5 6.7 7.0 7.2 7.5 7.7 8.0 8.2 8.5 8.7 9.0
X axis: Time in Seconds
Arc 135°

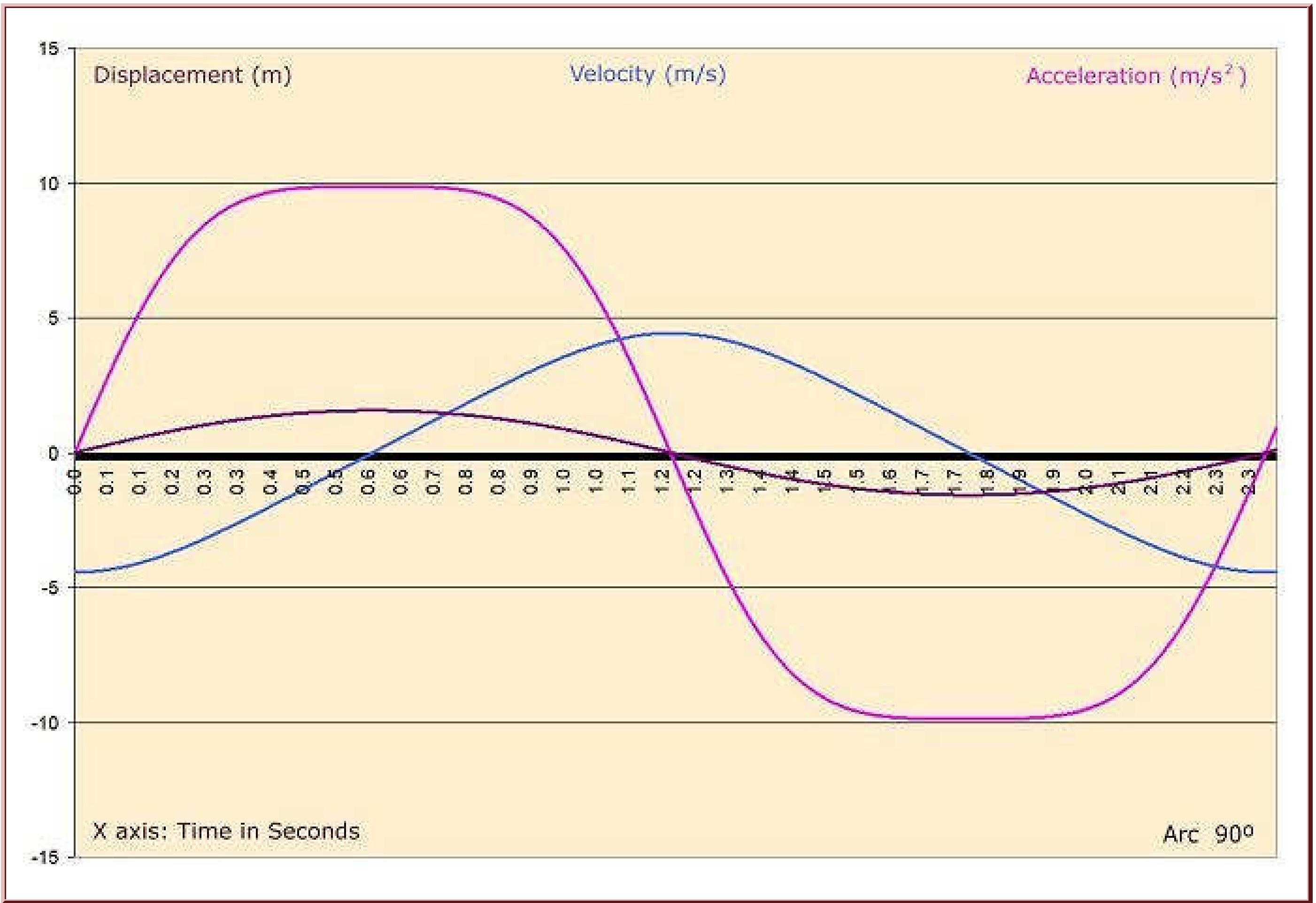


And around and around we go:



Looking more closely at the graph, we can see that the period is 2 seconds (2.00664) when the arc is 13°.

However, graph shows us that the period is 2.3 seconds (2.3532) when the arc is 90°. It takes longer for the pendulum to swing over and back.

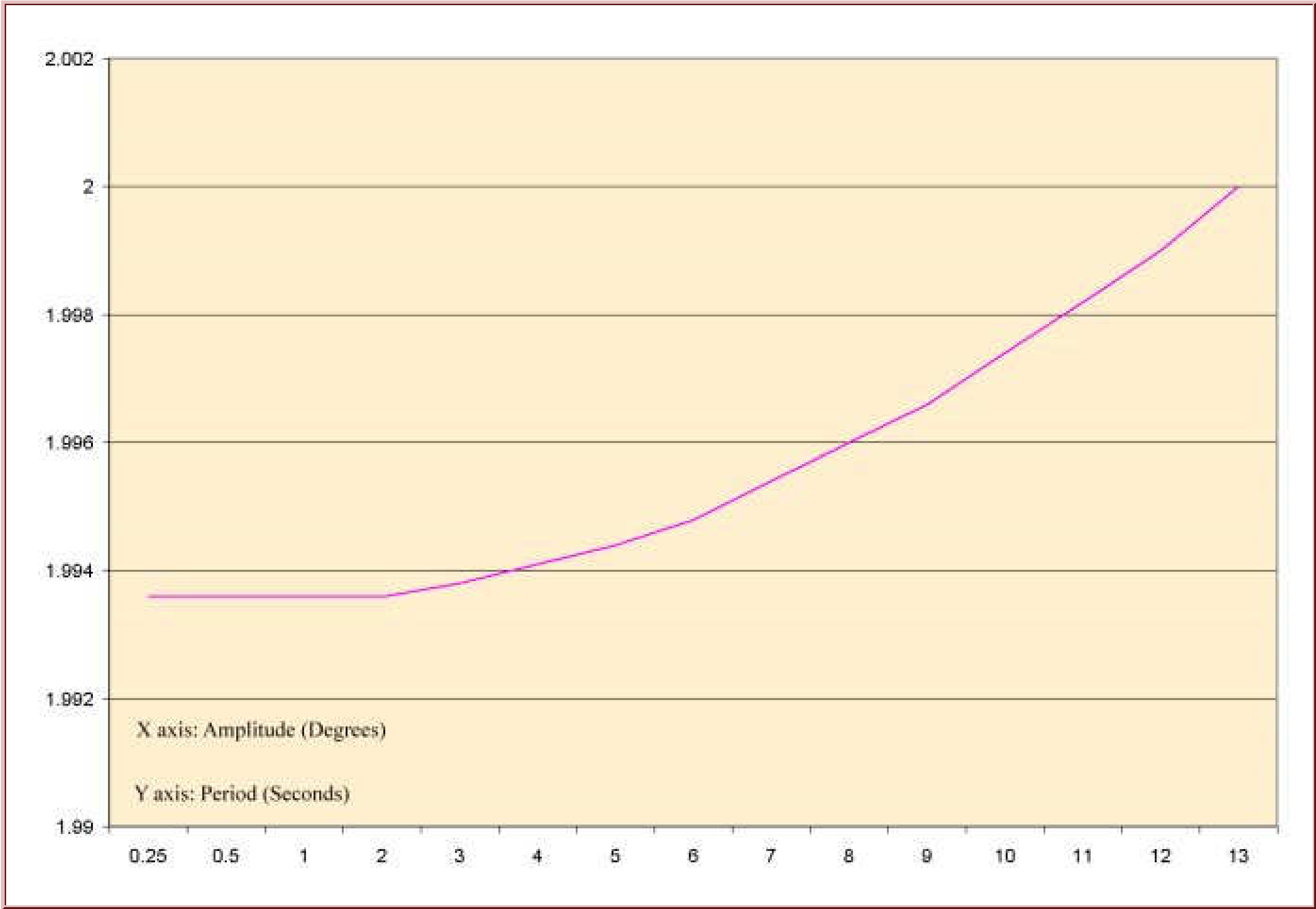


The Cause of Circular Error:

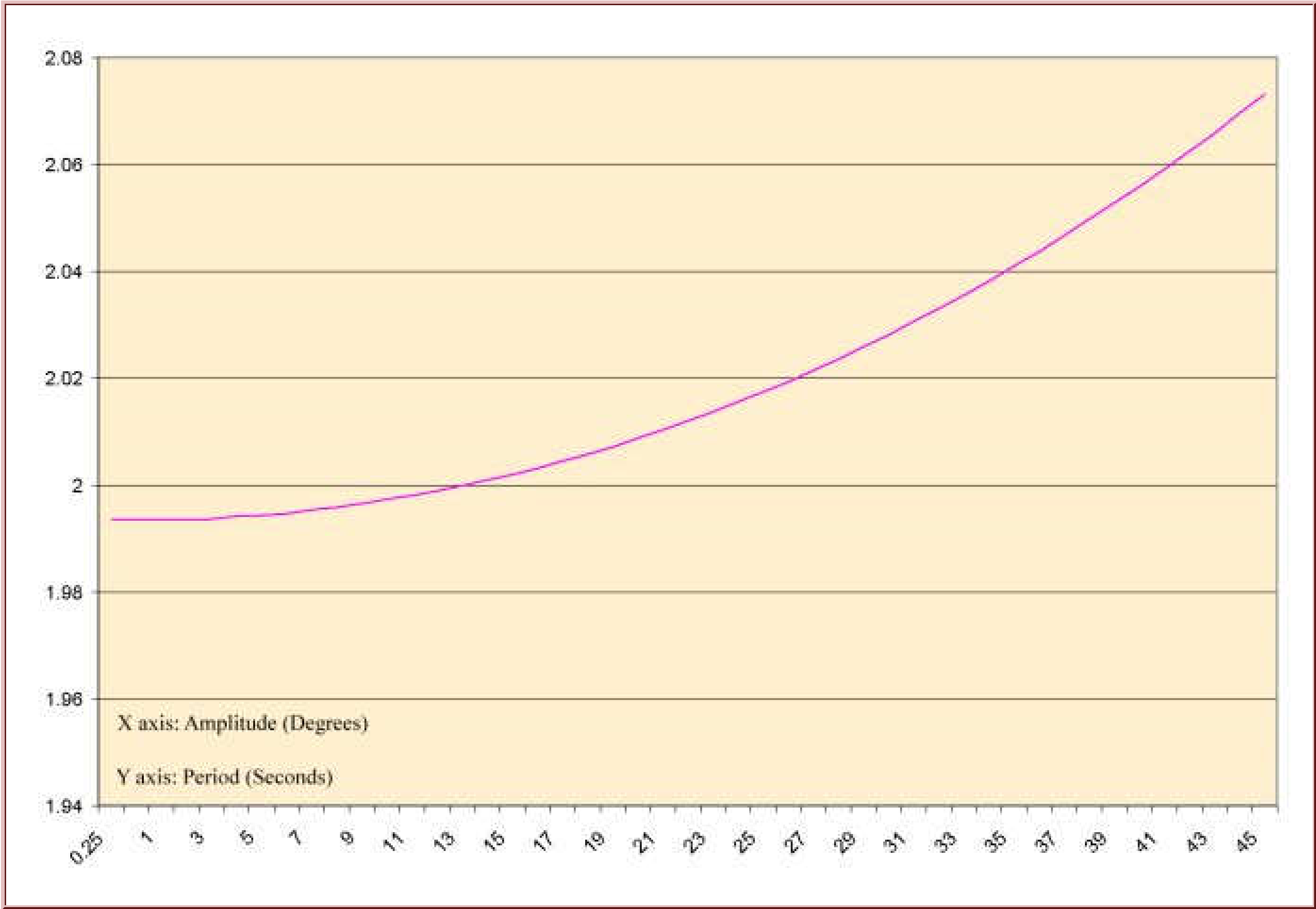
When the angle is doubled, the pendulum swings twice as far, but the acceleration does not double, and so the speed does not double either. If the pendulum swings twice as far and the speed does not double, then it takes <u>longer</u> to reach the other side: the period increases. The change in the period with changing angle is called circular error.

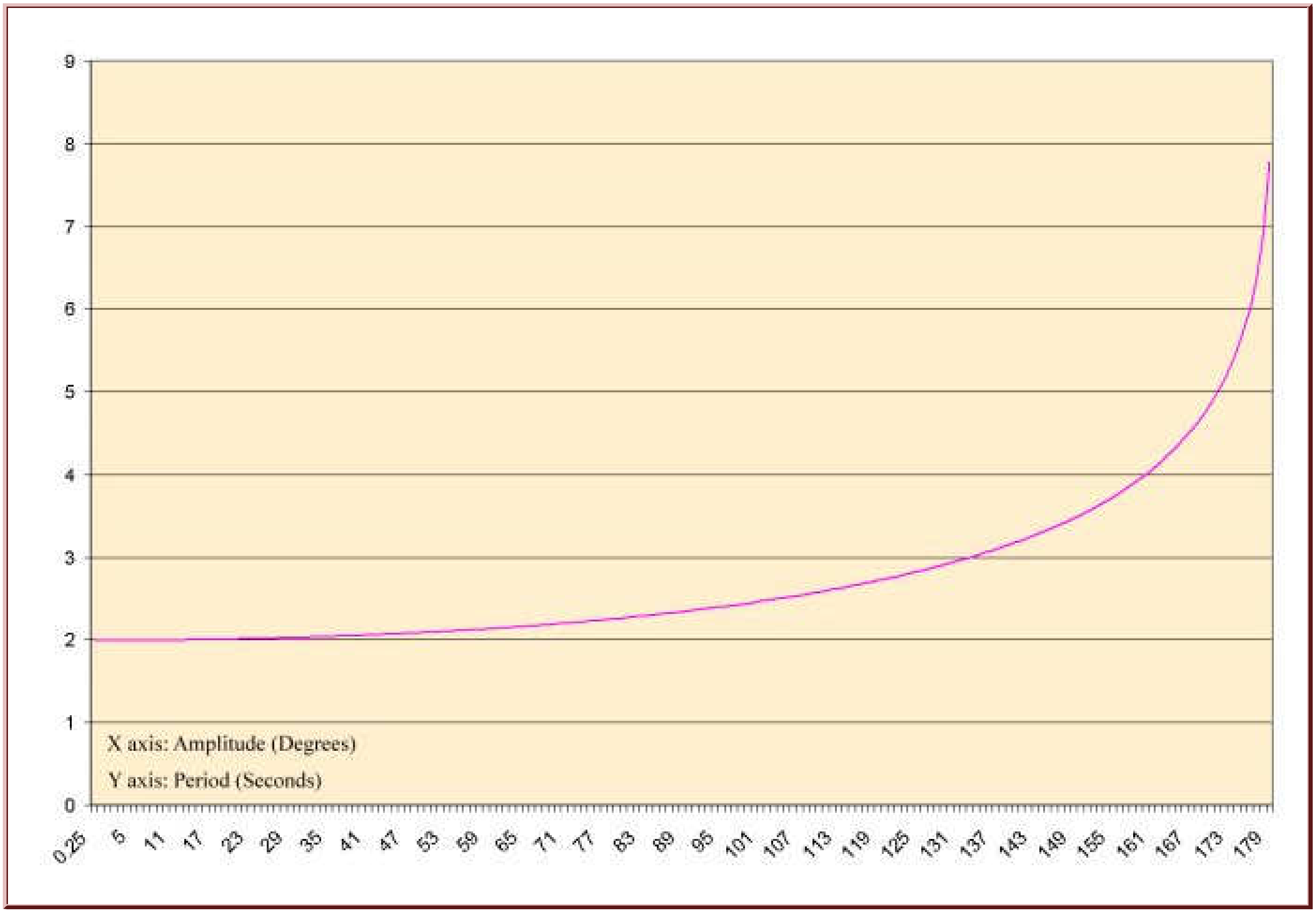
Note that I am referring to peak angular acceleration and peak angular velocity here.

The graphs below show how the period increases rapidly (exponentially) as the angle increases. The period is the time it takes to go back and forth, which is two seconds in this example. The curves are not perfectly smooth because of small rounding errors in the data.



The graph shows that the period reaches two seconds when the angle reaches 13°. This means that the period is less than two seconds for smaller angles, causing a clock to gain time, but greater than two seconds for larger angles, causing a clock to lose time as a result of circular error. Notice that, for small angles, of 3° or less, the curve becomes flat because of what is called Linearization (when sinθ is approximately equal to θ).


2.08
2.06
2.04
2.02
2
1.98
1.96
1.94
X axis: Amplitude (Degrees)
Y axis: Period (Seconds)
0.25
1
3
5
7
9
11
13
15
17
19
21
23
25
27
29
31
33
35
37
39
41
43
45

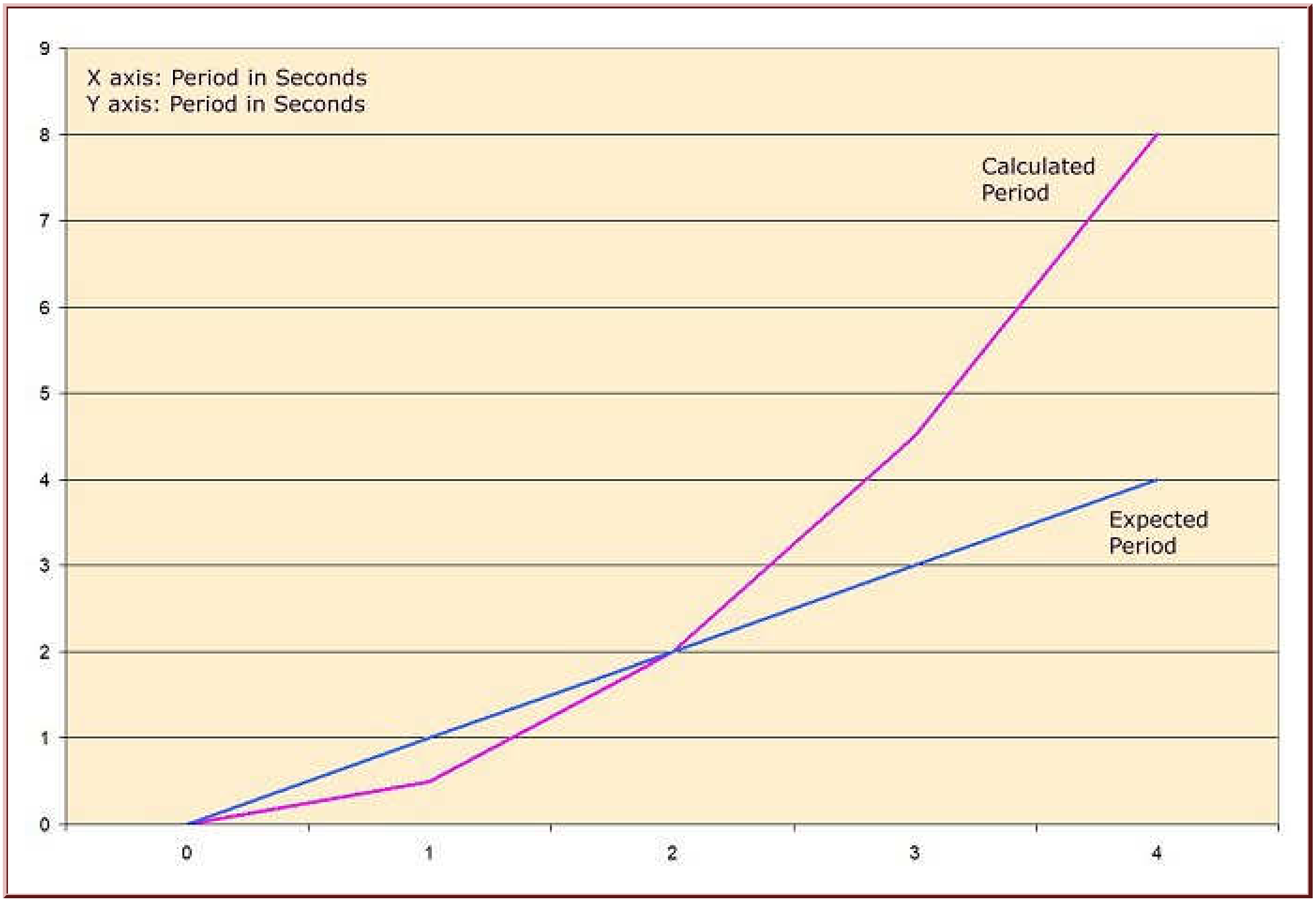


Changing the Length:

Performing similar calculations by changing the length of the pendulum, but keeping the amplitude unchanged at 13°, shows that the pendulum length has a stronger effect upon circular error. In the chart below, the left column shows the expected period (time taken to go back and forth) in seconds. The next column shows the pendulum length in metres for the corresponding period. The right column shows the calculated period, using the spreadsheet at the top of this page.

period	length	time
0.25	0.015525	0.0314
0.5	0.062101	0.3752
1	0.248405	0.5002
2	0.993621	2.002
3	2.235648	4.5008
4	3.974486	8.0008

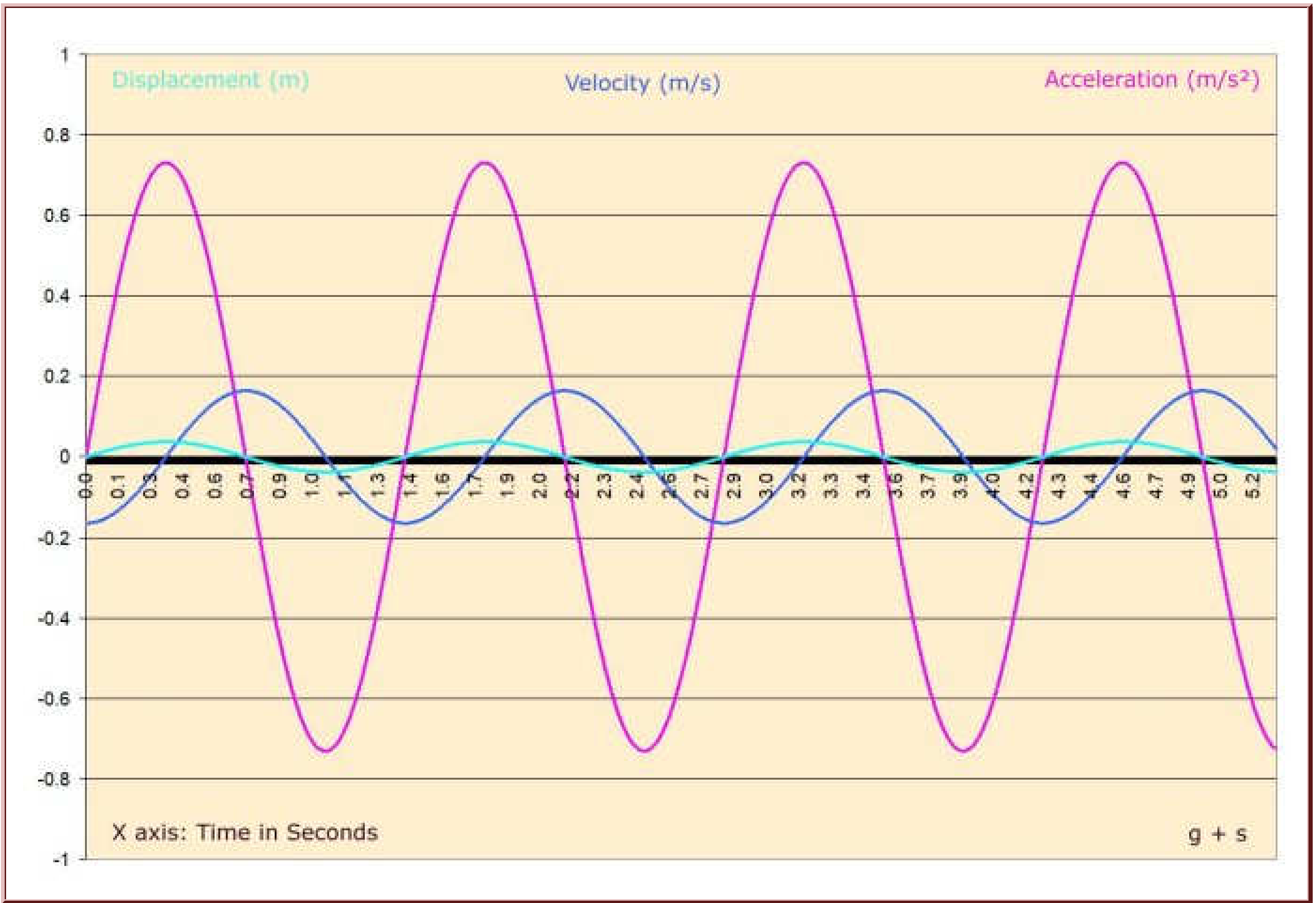
The graph below shows the expected period on the horizontal line (X axis) and the calculated period on the vertical line (Y axis). The data compares the one-second pendulum, which has an expected period and a calculated period of 2 seconds, with the periods for a pendulum of other lengths. The blue line shows what the line would look like without circular error (Y=X).



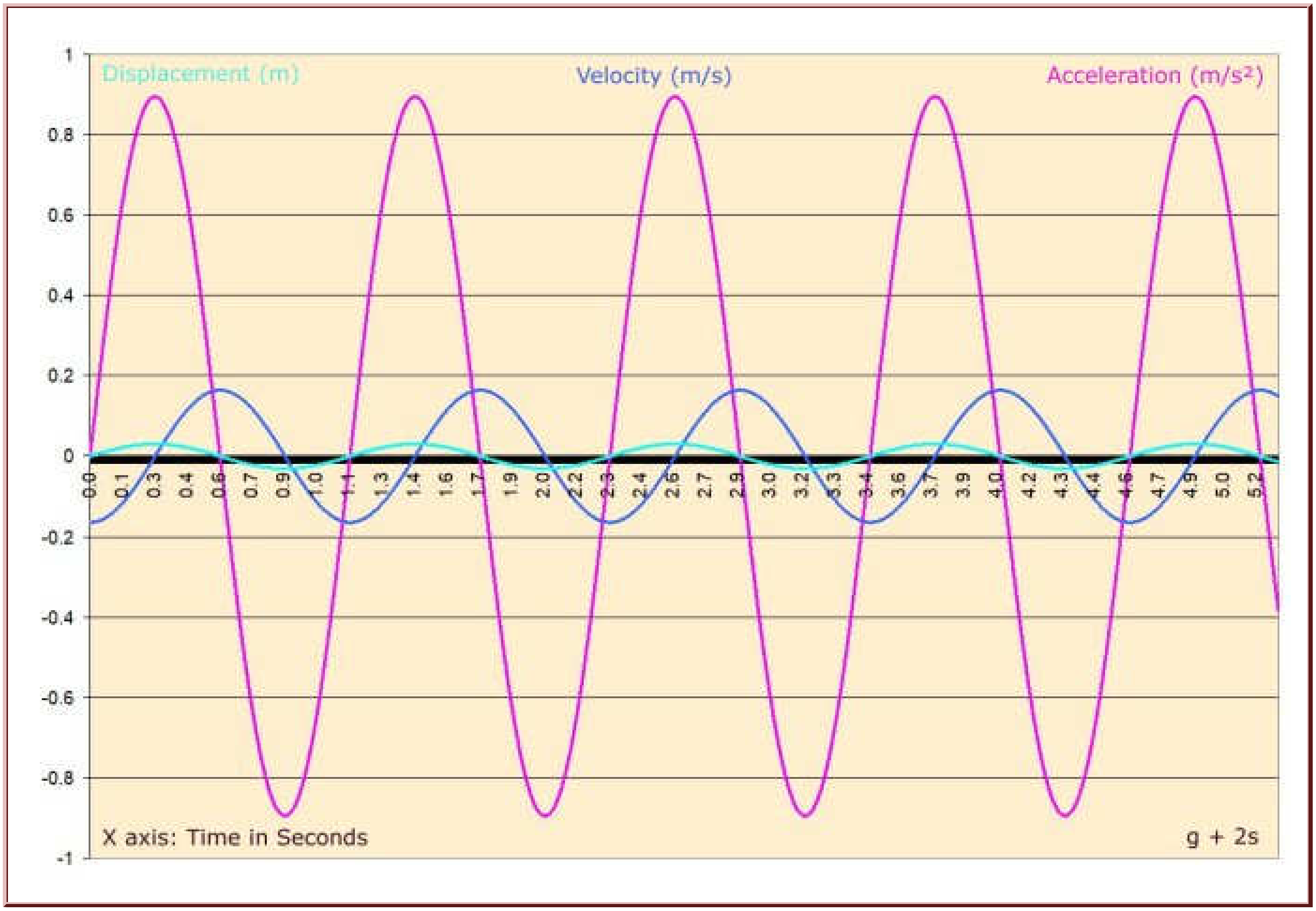
Adding a Suspension Spring:

Looking at the same one-second pendulum and adding a suspension spring, we get new graphs. The coefficient of elasticity of a spring deflected by bending remains constant, which is more or less true for small angles of 3° or less. In this example, I assume that the initial force applied by the suspension spring is equal to the initial force applied by gravity to bring the pendulum back to a vertical position, and the two forces are added together. Looking at a pendulum with an arc of only 3°, the graph for gravity looks like this one.

When a suspension spring is added in this simulation, the acceleration increases, the velocity remains the same, the angle decreases, and the period decreases.



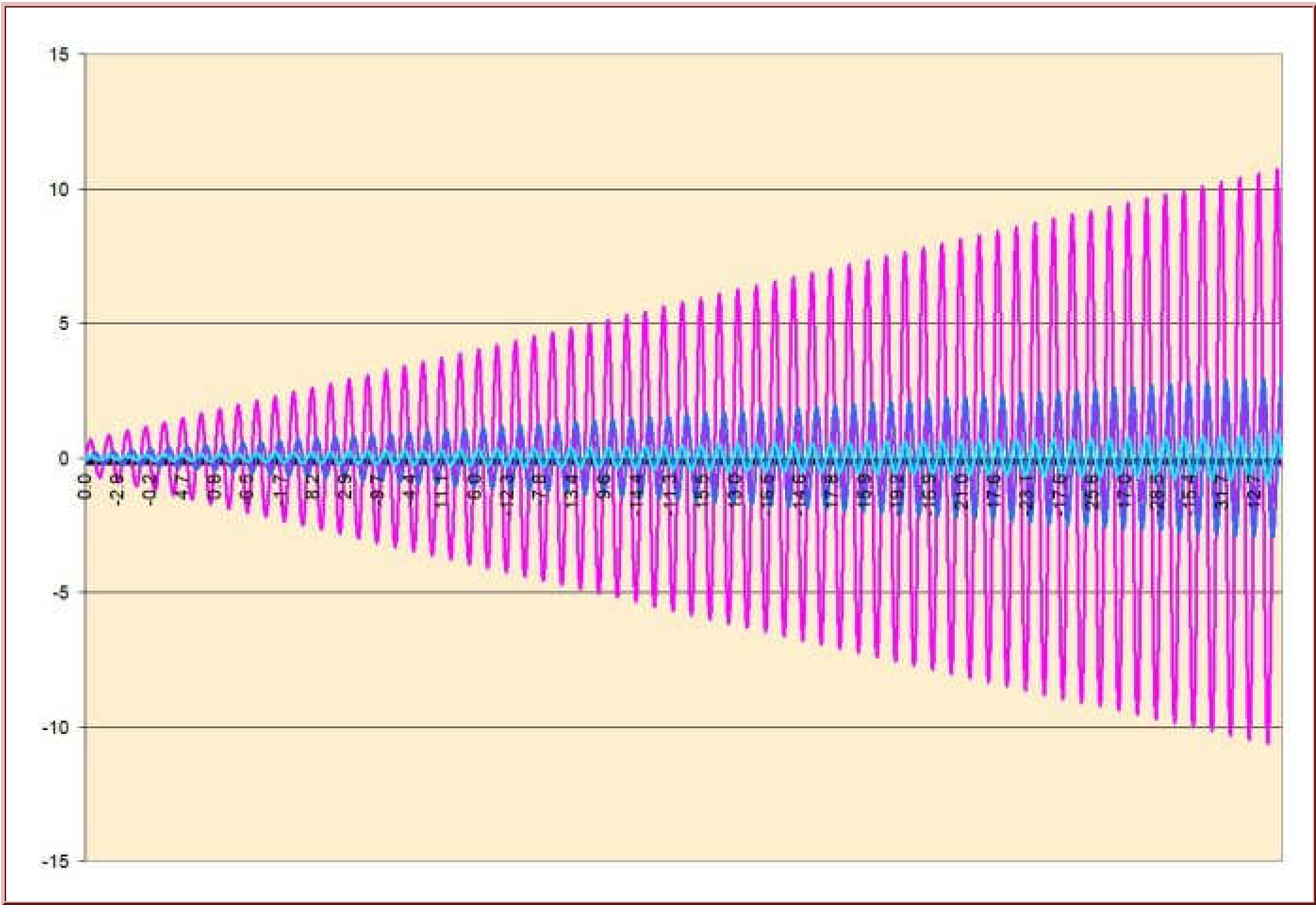
When a stronger suspension spring is added, the acceleration increases further, the velocity remains the same, the angle decreases further, and the period decreases further.



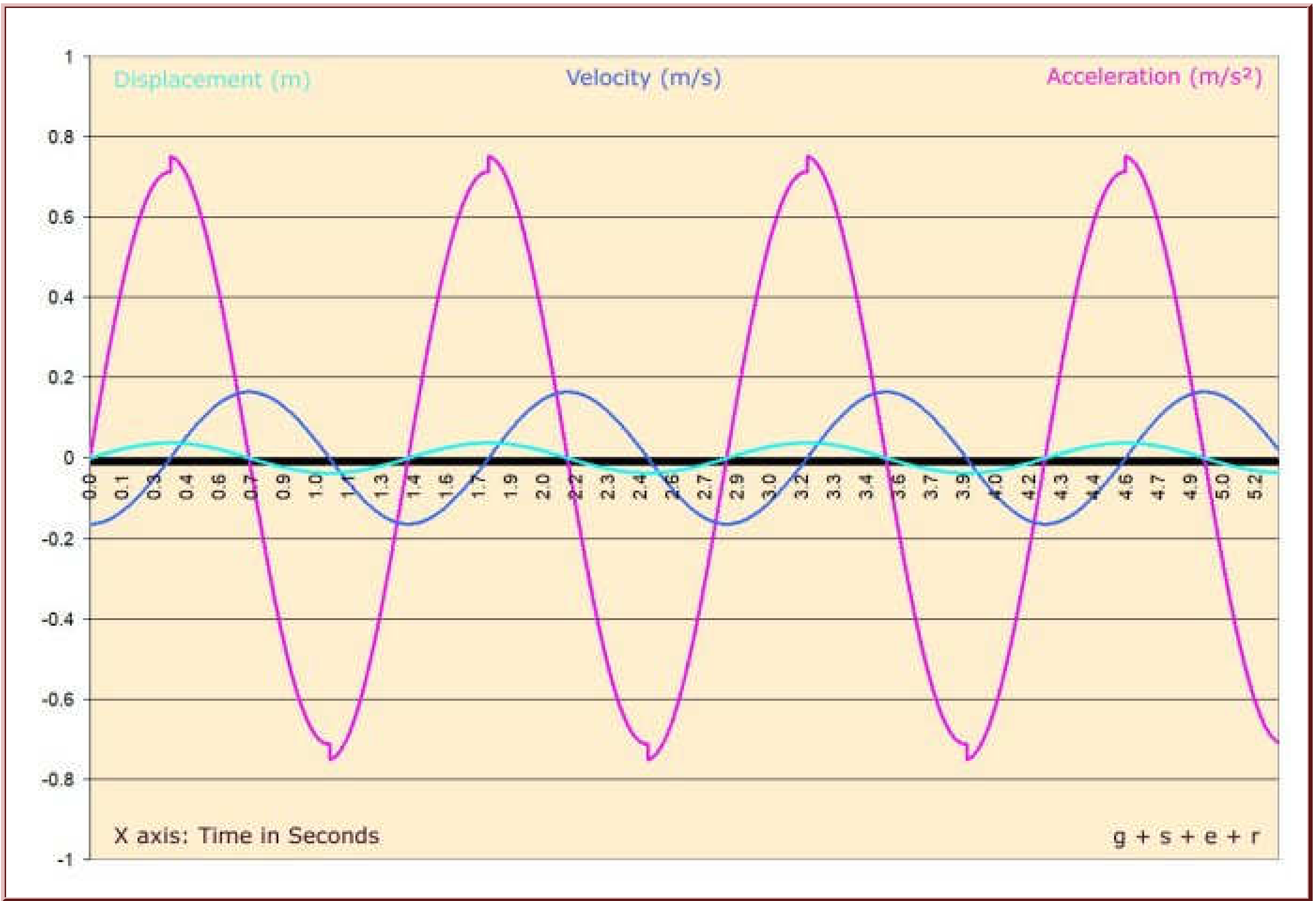
	G	G+S	G+2S	
Acceleration	0.517	0.731	0.894	m/s²
Velocity	0.164	0.164	0.164	m/s
Angle	3.000	2.122	1.730	degrees
Period	1.994	1.410	1.150	seconds

Adding an Escapement:

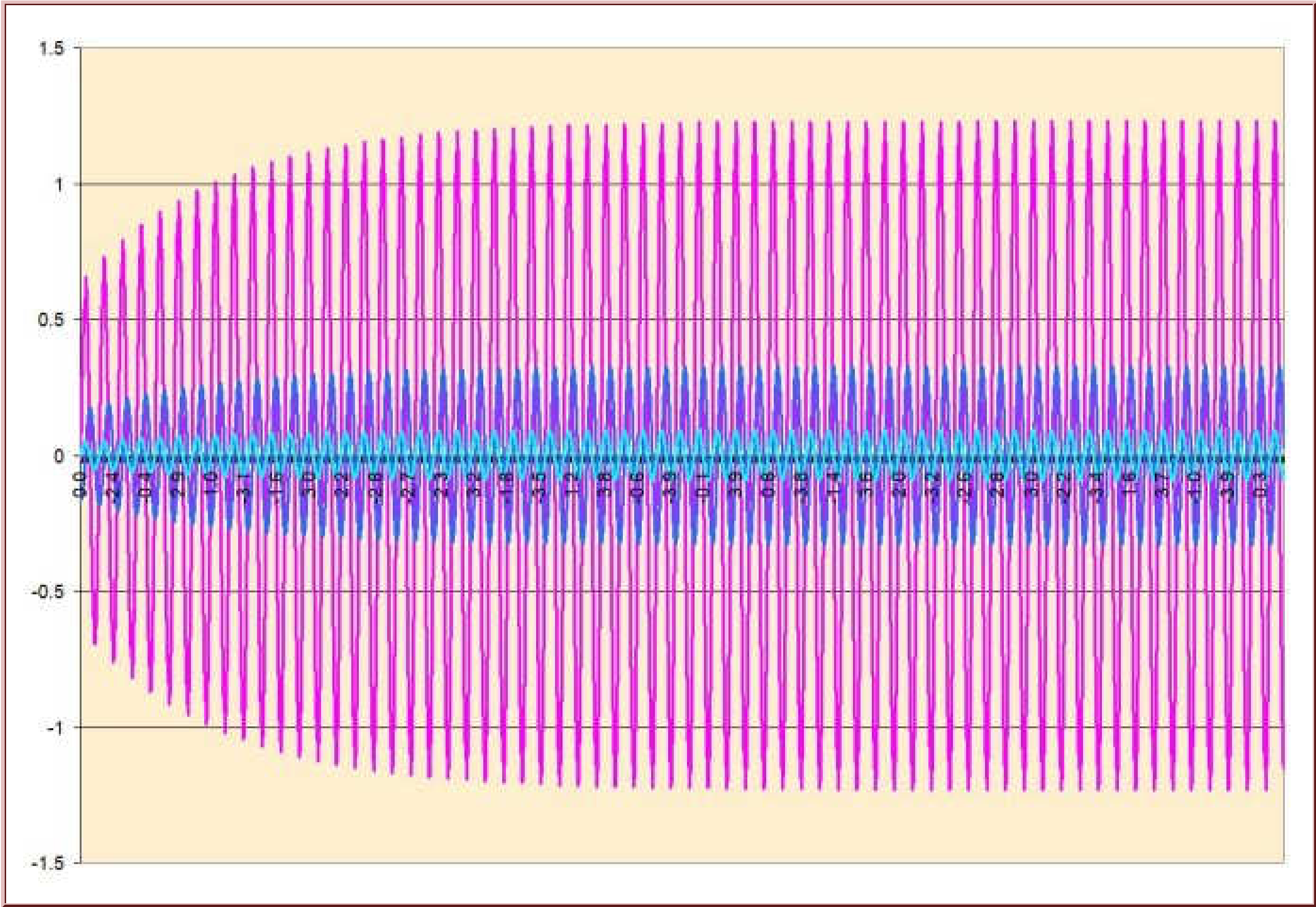
When an escapement is added to keep a pendulum going, as in a clock, the acceleration increases because there are now three forces acting upon the pendulum: the force of gravity, which is a function of sinθ, the elastic force from the suspension spring, which is a function of θ, and the force from the Graham escapement, as the escape wheel's tooth slides across the pallet's impulse face. In this example, the pallet has an arc of 6° and the pendulum has an arc of 3°, so the peak acceleration from the escapement that reaches the pendulum is multiplied by cos2θ, which has a range of 0.997 and 1 when the arc is 6°. The force of the escapement changes direction when the velocity of the pendulum changes direction. Acceleration gets out of control.



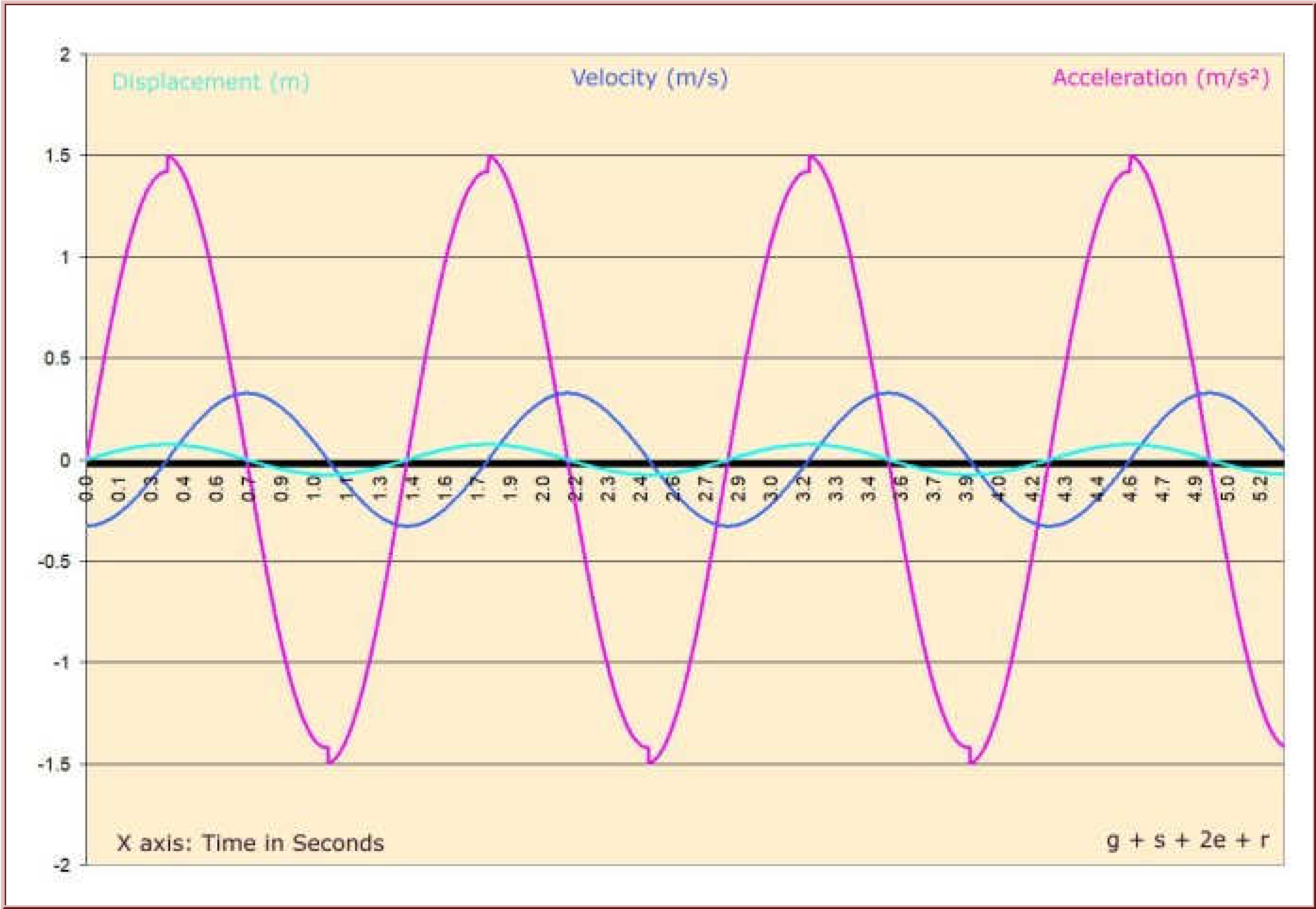
Therefore, an energy loss (resistance) needs to be introduced into the equation, such that the energy gains from the escapement are equal to the energy losses from air resistance, the bending of the suspension spring, and so on. The result makes for an interesting graph.



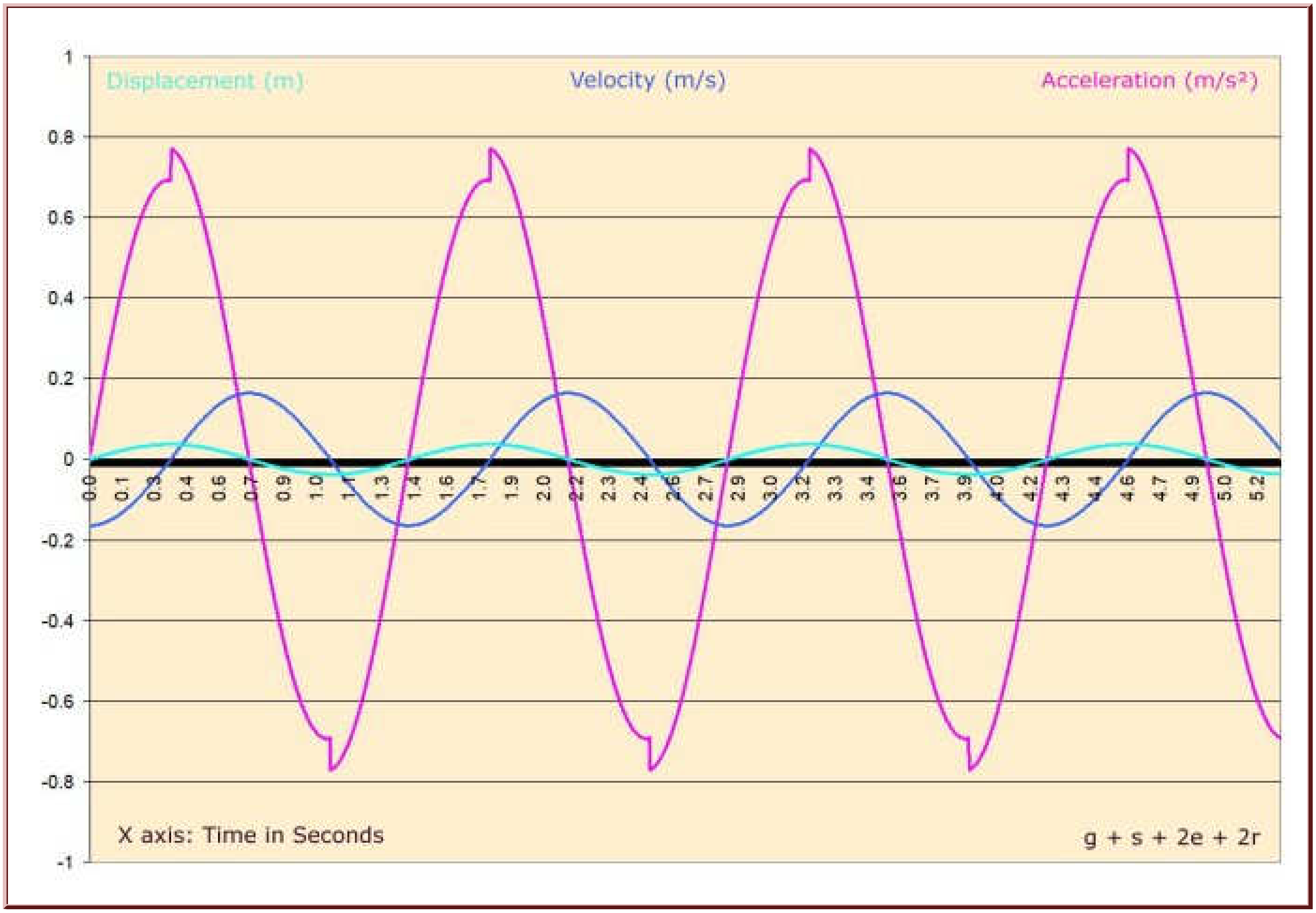
When the energy from the escapement is increased, the acceleration, the velocity, and the angle increase, until an angle is reached when the energy losses become equal to the energy gains from the escapement, and an equilibrium is reached.



The amplitudes of the curves are greater at equilibrium. The period remains unchanged.



However, when the energy losses are also increased in proportion to energy gains in this simulation, the amplitudes and the period remain unchanged.



	G	G+S	G+S+E+R	G+S+2E+R	G+S+2E+2R	
Acceleration	0.517	0.731	0.751	1.497	0.771	m/s²
Velocity	0.164	0.164	0.164	0.327	0.164	m/s
Angle	3.000	2.122	2.122	4.233	2.122	degrees
Period	1.994	1.409	1.409	1.409	1.409	seconds

From the point of view of clock repair, the simulation offers insight into the forces that act upon a pendulum and affect timekeeping accuracy.

1. If you use a stronger suspension spring, the period will decrease significantly and the clock will gain time. The angle will also decrease, so a stronger mainspring or a heavier weight may be needed to keep the clock running.

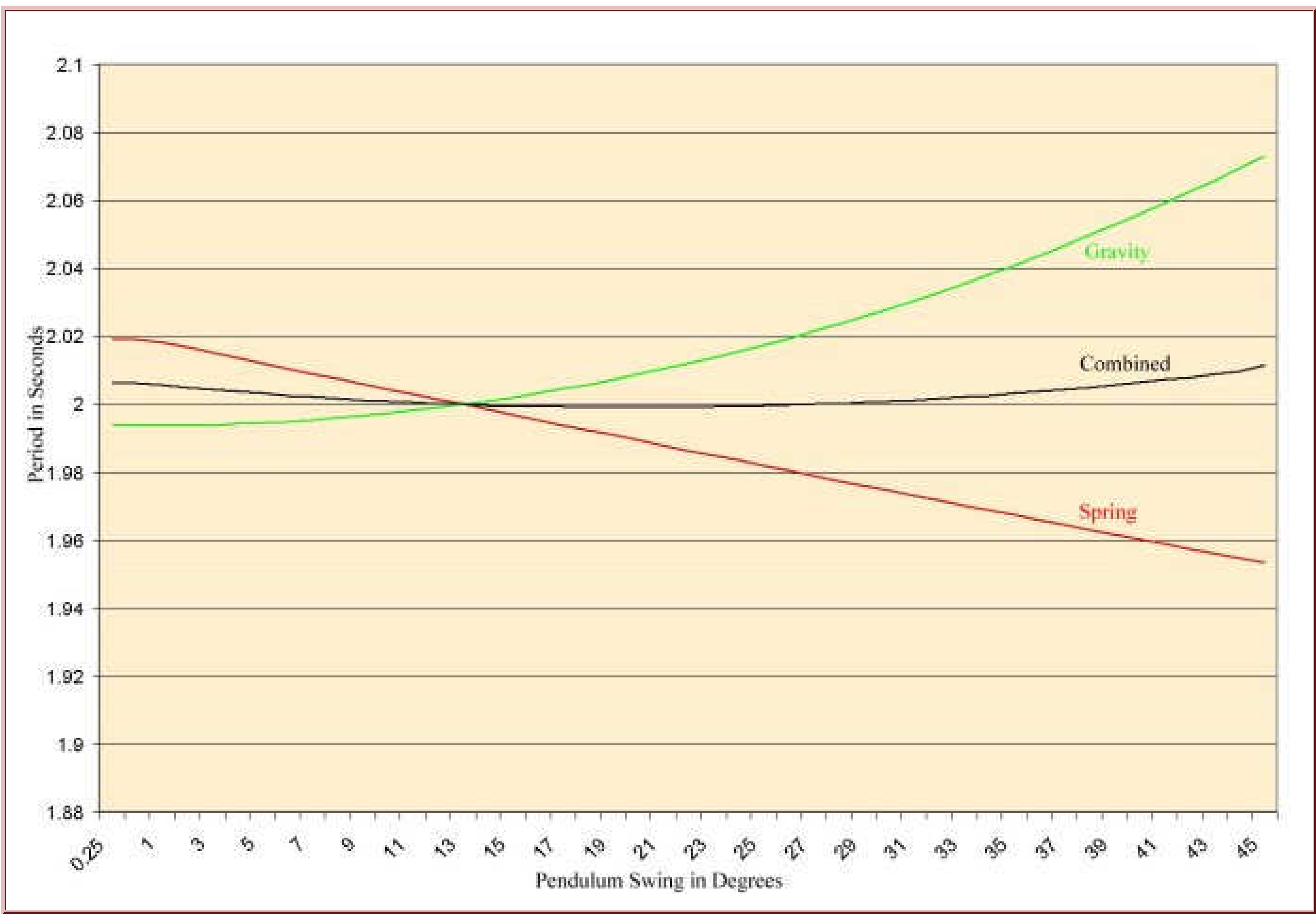
2. The simulation above suggests that increasing the energy from the escapement would not affect the period. In practice, however, I have found

that using a stronger mainspring or a heavier weight will decrease the period slightly, so the clock will gain time.

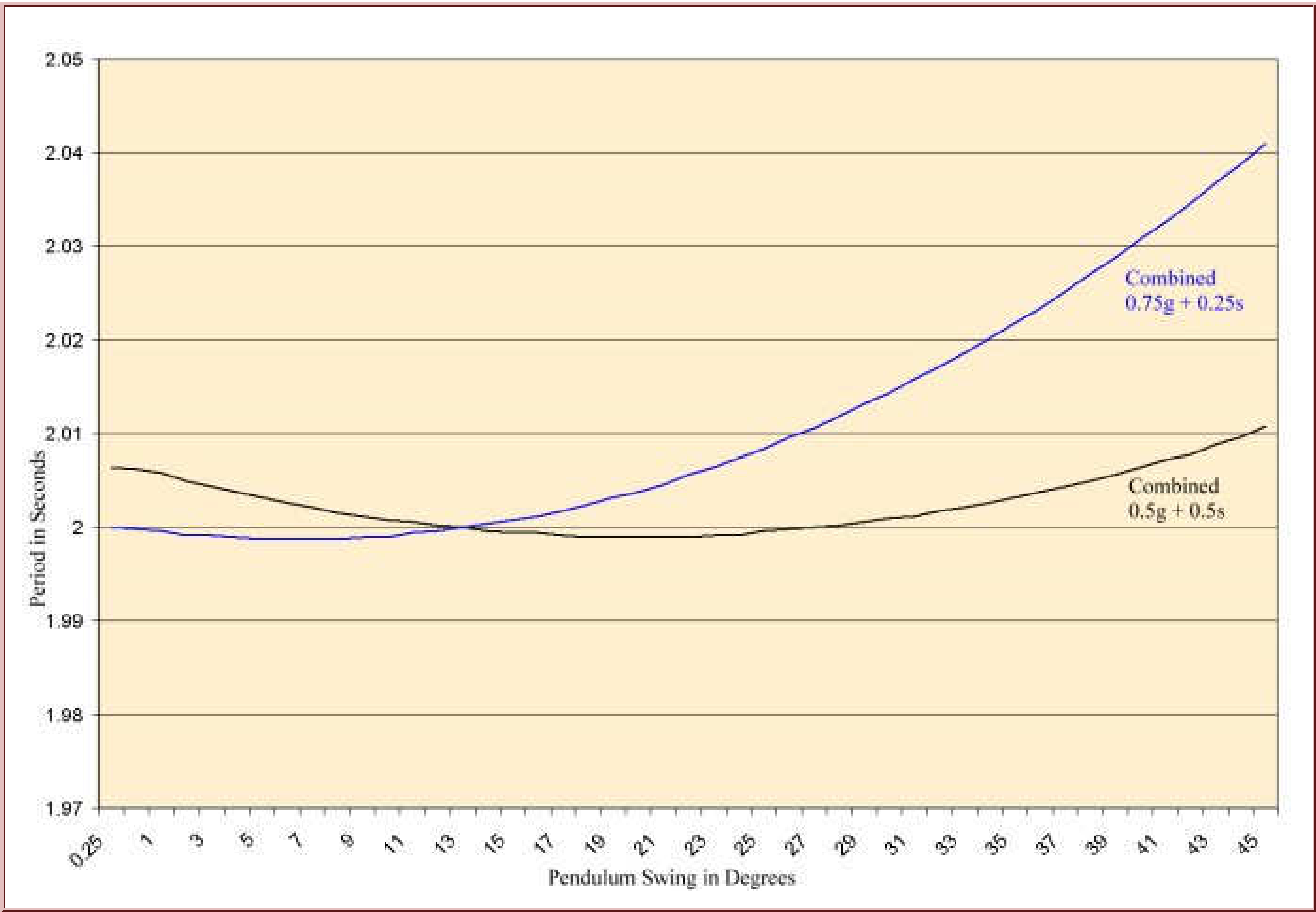
The clock will gain time because the stiffness of the suspension spring becomes ever so slightly more stiff as the spring is bent further, where the coefficient of elasticity is expressed by $k = k_1 + k_2(\theta)$. The values for k_1 and k_2 depend on the properties and the composition of the metal used in the suspension spring. Acceleration could be expressed by:

$$k(\theta) = k_1(\theta) + k_2(\theta^2)(\mathrm{sign}(\theta))$$

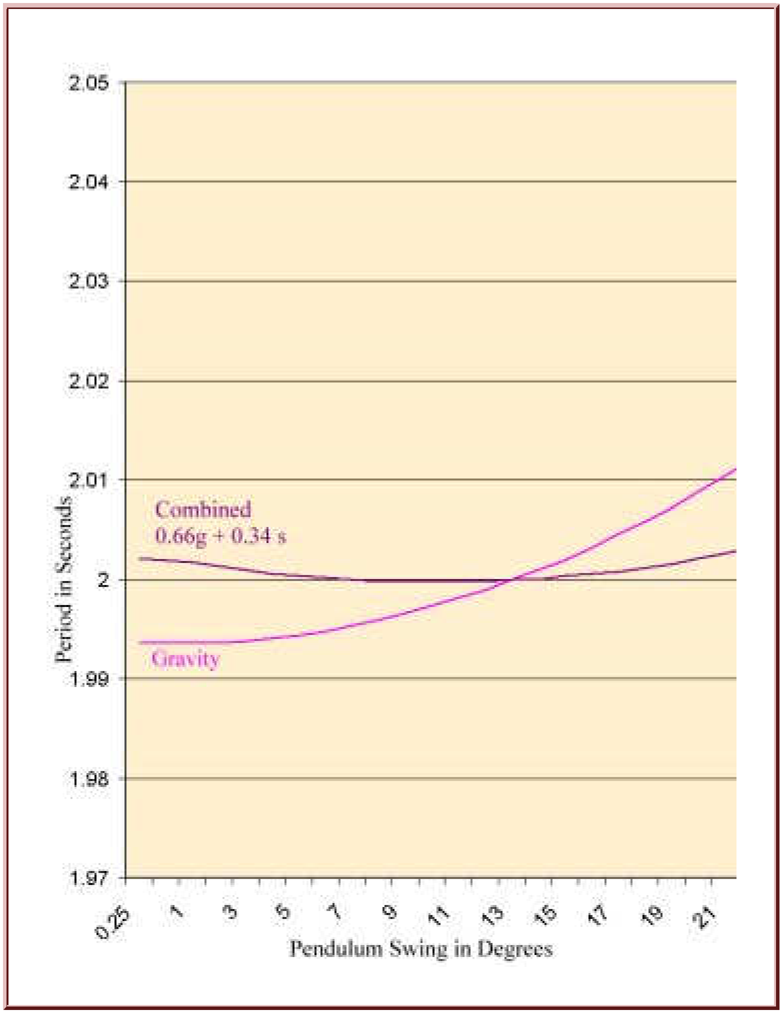
The same spreadsheet could be used to perform a numerical integration, as was used to calculate circular error above, to produce the following graph. The simulation in this graph shows how, for smaller angles, the suspension spring causes a clock to gain more time than the effect of gravity causes the clock to lose time, producing a net time gain as the angle of pendulum swing increases.



The solution to the problem is really quite simple. If you install a stronger mainspring and the clock gains a lot of time, then the suspension spring is too strong and you need to install a weaker suspension spring, (assuming the clock has a Graham escapement):



The time gained because of the suspension spring could be reduced in a typical pendulum clock by choosing a thinner suspension spring, and making the spring a little narrower to reduce its strength further if needed, because most suspension springs are stronger than necessary. The time gain caused by the suspension spring would cancel out with the time lost because of circular error: in this simulation, the circular error affecting timekeeping accuracy would be negligible for pendulum swings between 5 and 15 degrees from the vertical position. Make your clock great again.



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